ALL ARTICLES in ARXIVE (1992-1)

EDITE and REVIEW

JASEM HAMOUD

PART 1

This book is a simple part of a long-term task in order to revise the articles and integrate them into a book that includes various articles from various

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Chapter 1

Introduction.

Discrete Mathematics via Logic and Proof is a comprehensive book that delves into the fundamental concepts and principles of discrete mathematics. This book is designed to provide a solid foundation in discrete mathematics, focusing on the logical and proof-based aspects of the subject. The book is structured to cater to students and researchers who are interested in understanding the theoretical and practical applications of discrete mathematics in various fields.

This book consists of a set of articles that were published on the **arxive** website, as I edited them and put them in a book in order to benefit from this book.this idea came during my research on some important articles on this site and I decided that this would be the first part.

Which led me to consider that this is the first part is the following:

1. the presence of a number of references that were placed in the articles and which were not quoted or referenced in the research, which made me delete these references.
2. The existence of some errors in some articles that I corrected and did not point out so as not to detract from the value of the article, and perhaps the researcher did not pay attention to this, as happened in the references.
3. Arrange the articles as arranged on the site and try to organize the texts as much as possible so that the text and the article are largely identical.
4. Many of the articles have appendices, and these appendices are integrated texts and other articles published in other editions of arxive, so we considered this section to be the first part of the book.

My work in this book is that I rewrote the articles in a new format and identical to the original and collected the articles in an orderly manner as stated on the site and corrected some errors and deleted superfluous things such as some references that were not used but were placed in the article.

This book is a simple part of a long-term task in order to revise the articles and integrate them into a book that includes various articles from various branches.

Everything that has been written in mathematics and what will be written, we have the right to provide criticism to it and put our notes on it, even if the work is complete, because mathematics is the language of results and not the language of writing only, and from this section we put this part to you.

For any feedback or suggestion please email us: *jasem1994hamoud@gmail.com*

The book can be citition as follows:

Hamoud, J., ALL ARTICLES in ARXIVE (1992-1)
EDITE and REVIEW, Name of author article, name of article 2024,

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By:
Jasem Hamoud Moscow- Russia 21.05.2024

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Chapter 2

First 10^{th} Articles

2.1 On the removable singularities for meromorphic mappings

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Abstract:
If E is a nonempty closed subset of the locally finite Hausdorff (2n-2)-measure on an n-dimensional complex manifold Ω and all points of E are nonremovable for a meromorphic mapping of $\Omega \setminus E$ into a compact Kähler manifold, then E is a pure (n-1)-dimensional complex analytic subset of Ω.
keywords: meromorphic continuation, removable singularities, pseudoconcave sets.
subjclass Primary: 32D15, 32D20; Secondary 32D10.

2.1.1 Introduction.

This paper was inspired by the following question of E.L.Stout [75] Let E be a closed subset of the complex projective space $\mathcal{P}^n$ $(n \geq 2)$ such that the Hausdorff $(2n-2)$-measure of E (with respect to the Fubini-Study metric) is less then that of any complex hyperplane of $\mathcal{P}^n$. If it is true that E is then a removable singularity for meromorphic functions? Using natural projections of $\mathcal{P}^n$ onto hyperplanes G.Lupacciolu [76] has shown the removability of E under additional conditions on the sizes of E and a maximal ball in the complement. (The projection of $\mathcal{P}^n$ onto a hyperplane does not decrease Hausdorff measures, as it take place in the Euclidean space.) Using an Oka–Nishino theorem [77] on pseudoconcave sets we prove here the following.

Theorem 2.1.1. *Let E be a closed subset of the locally finite Hausdorff $(2n-2)$-measure on an n-dimensional complex manifold Ω and let f be a meromorphic mapping of $\Omega \setminus E$ into a complex manifold X. If X has the meromorphic extension property and E does not contain any $(n-1)$-dimensional closed analytic subset of Ω then f is continued to a meromorphic mapping of Ω into X.*

Here we say that X has the meromorphic extension property, if any meromorphic map

$\varphi : T \to X$ of the "squeezed polydisc"

$$T = (z, w) \in \mathbb{C}_z^{n-1} \times \mathbb{C}_w : |z| < r, |w| < 1 \text{or} |z| < 1, 1 - r < |w| < 1,$$

$0 < r < 1$, $n \geq 2$, extends to a meromorphic map $\tilde{\varphi} : U \to X$ of the unit polydisc $U : |z| < 1, |w| < 1$. By a recent result of S.M.Ivashkovich [78] every compact Kähler manifold X has the meromorphic extension property, so we have a lot of nice examples of such X. The case of meromorphic functions ($X = \mathbb{P}^1$) is almost trivial in the consideration: every meromorphic function in a squeezed polydisc is represented as a ratio of two holomorphic functions (see [80]) and thus it is meromorphically continued into U. So the answer on the question of Stout is positive because the mentioned E can not contain any complex analytic subset of $\mathbb{P}^n$ (by a Chow theorem such a set is algebraic, and the Hausdorff $(2n-2)$-measure of it is not less then the measure of a complex hyperplane. As pointed me E.L.Stout, this question was solved already by his student Mark Lawrence by a similar method.

2.1.2 Results

A closed subset Σ of a complex manifold is called locally pseudoconcave if for every point $a \in \Sigma$ there exists a Stein neighbourhood V such that $V \setminus \Sigma$ is Stein. Let E' be the set of points $a \in E$ such that f meromorphically extends into a neighbourhood of a. Then $S := E \setminus E'$ is closed. As the complement to E is locally connected in Ω, these local meromorphic continuations of f in points of E' glue together into the unique meromorphic map $f : \Omega \setminus S \to X$ (we preserve the notation f). The proof of the Theorem is accomplished now in two steps. Firstly we prove that S is locally pseudoconcave (Lemma 1), and secondly we prove that S is complex analytic (Lemma 2).

Lemma 2.1.2. *S is locally pseudoconvex in Ω.*

Proof. Let $a \in S$, $V \ni a$ is biholomorphic to a ball in $\mathbb{C}^n$ and $\varphi : T \to V \setminus S$ is a holomorphic embedding. Then (V is Stein) φ extends to a holomorphic embedding $\tilde{\varphi} : U \to V$ (see [78]). As X has the meromorphic extension property, the meromorphic map $f \circ \varphi : T \to X$ extends to a meromorphic map of U into X and thus f meromorphically continues into the domain $\tilde{\varphi}(U) \subset V$. By the definition of S, $\tilde{\varphi}(U)$ does not intersect S, and thus we have proved that $V \setminus S$ sutisfies the condition "p_7-convexity" of Docquier–Grauert [78]. It follows that $V \setminus S$ is Stein. □

Lemma 2.1.3. *Let S be a nonempty locally pseudoconcave subset of the finite Hausdorff $(2n-2)$-measure on an n-dimensional complex manifold Ω. Then S is a pure $(n-1)$-dimensional complex analytic subset of Ω.*

Proof. The statement is local, so we can assume that Ω is a domain in $\mathbb{C}^n$. Let $a \in S$ and coordinates (z, w), $z = (z_1, ..., z_{n-1})$ in $\mathbb{C}^n$ are choosen in such a way that $a = 0$ and the set $S \cap \{z = 0\}$ is finite or countable (it can be done obviously). Then there exists a neighbourhood $V = V' \times V_n \ni 0$ such that the projection of $S \cap V$ into V' is proper. It follows that fibres $S \cap V \cap \{z = c\}$ are finite for almost every $c \in V'$. By an Oka–Nishino theorem [43] $S \cap V$ is then a complex analytic set of pure dimension $n - 1$. □

I would like to thank G.Lupacciolu for the reprint of his paper [42], which stimulated this work.

2.2 Factorizations of natural embeddings of l_p^n into L_r, II

T. Figiel, W. B. Johnson* and G. Schechtman†

2.2.1 Introduction

In this continuation of [1], we show that in some situations considered in [1], conclusions of certain theorems can be strengthened. More explicitly, suppose that T is an operator from some Banach space into L_1 which factors through some L_1-space Z as uw and normalized so that $\|w\| = 1$. In **Corollary 12.A** we show that if T is the inclusion mapping from a “natural" n-dimensional Hilbertian subspace of L_1 into L_1, then u well-preserves a copy of l_1^k with k exponential in n (where “well" and the base of the exponent depend on $\|u\|$ and on a quantitative measure of “naturalness"). This improves the result of [1] that the same hypotheses yield that l_1^k well-embeds into uZ. Corollary 12.B gives a similar improvement of **Corollary 1.5** in [1]; that is, Corollary 12.B is the same as **Corollary 12.A** except that the operator T is assumed to be a mapping from a space whose dual has controlled cotype into L_1 which acts like a quotient mapping relative to a “natural" Hilbertian subspace of L_1.

Corollary 20 strengthens the conclusion of **Proposition 4.3** in [1] in a similar manner; it states that an operator from a $C(K)$ space which well-preserves a copy of l_2^n also well-preserves a copy of l_∞^k with k exponential in n (rather than just have rank which is exponential in n). This can be viewed as a finite dimensional analogue of a particular case of a result of *Peczy′nski* [[9]] stating that every non weakly compact operator from a $C(K)$ space preserves a copy of c_0.

In Theorem 2.2.22 we apply the earlier results in order to prove that for each m there is an m-dimensional normed space G such that any superspace of G with a good unconditional basis must contain a copy of l_∞^k with k exponential in m.

We thank J. Bourgain for pointing us in the right direction on the material presented here. After proving the results in [1], we suggested to him that there might be a translation invariant operator T of bounded norm on L_1 of the circle which is the identity on the span of the first n Rademacher functions and which does not preserve l_1^k with k exponential in n. By disproving this conjecture, Bourgain started us thinking that **Corollary 12.A** was true.

2.2.2 Preliminaries, a quantitative version of Rosenthal's lemma

In this section we prove, in Proposition 1 below, a quantitative version of Maurey's formulation [8] of Rosenthal's lemma [3] stating that an operator into $L_1(\mu)$ either factors through an $L_p(\nu)$ space for some $p > 1$ via a change of density or is of type no better than 1. Our approach is in fact close in spirit to Rosenthal's original argument which (unlike some later arguments) was basically quantitative in nature. Recall that for $1 \leq p < q \leq \infty$ and for an operator $u\colon Z \to L_p(\mu)$

$$C_{p,q}(u) = \inf\{\|h\|_s \|h^{-1}u\colon Z \to L_q\|\}$$

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*Supported in part by NSF DMS-87-03815

†Supported in part by the U.S.-Israel BSF

where the *inf* is over all changes of measure h; i.e., over all $0 < h \in L_s(\mu)$ where $\frac{1}{q} + \frac{1}{s} = \frac{1}{p}$. In the statement of 2.2.1 and elsewhere in this paper t^* denotes $\frac{t}{t-1}$.

Proposition 2.2.1. *Let* $1 < p < q \leq \infty$ *and* $\sigma = 1 - \frac{q^*}{p^*}$. *Let* $T\colon Z \to L_1(\mu)$, *where* μ *is a probability measure. If* $C_{1,p}(T) = K$ *and* $\|T\colon Z \to L_p(\mu)\| = \mathcal{C}K$, *then for some* $m > \left(\frac{K}{2^{2+\frac{1}{p}}\mathcal{C}\|T\|}\right)^{p^*}$ *there exist* $z_1, \dots, z_m$ *in the unit ball of* Z *and mutually disjoint measurable sets* $F_1, \dots, F_m$ *such that for* $i = 1, \dots, m$ *one has*

$$\|1_{F_i} Tz_i\|_1^{\sigma} \|1_{F_i} Tz_i\|_q^{1-\sigma} \geq 2^{-(\frac{1}{p}+1)} K. \tag{1}$$

For the proof we need two lemmas.

Lemma 2.2.2. *Let* $g \in L_1$ *with* $\|g\| \leq 1$. *Suppose* E *is a* μ*-measurable set,* $1 < p < q \leq \infty$ *and*

$$\|1_{\sim E}\, g\|_p > \kappa > 0.$$

Then there exists a measurable set F, $F \cap E = \emptyset$, *such that*

$$\mu(F) < \Big(\frac{2^{1/p}}{\kappa}\Big)^{p^*},$$

$$\|1_F\, g\|_1^{\sigma} \|1_F\, g\|_q^{1-\sigma} > 2^{-\frac{1}{p}} \kappa.$$

Proof. Without loss of generality we can assume that $g \geq 0$. Set $F = [g > \gamma] \sim E$, where $\gamma > 0$ is defined below. Observe that

$$\int_{\sim F} g^p \leq \gamma^{p-1} \int_{\sim F} g \leq \gamma^{p-1}$$

and hence, by Hölder's inequality,

$$\kappa^p - \gamma^{p-1} < \int_F g^p \leq \Big(\int_F g\Big)^{1-t} \Big(\int_F g^q\Big)^t,$$

where $t = (p-1)/(q-1)$. Since $\mu(F) < 1/\gamma$, we can fulfill both conditions of the lemma by choosing γ so that $\gamma^{p-1} = \frac{1}{2}\kappa^p$..

$\square$

Lemma 2.2.3. *Let* $T\colon Z \to L_1(\mu)$, μ *being a probability measure. Suppose* $1 < p < \infty$ *and*

$$C_{1,p}(T) = K, \qquad \|T\colon Z \to L_p(\mu)\| = \mathcal{C}K.$$

If $0 < \kappa < K$, $\eta \geq 0$ *and* $\mathcal{C}\eta^{1/p^*} \leq 1 - \kappa/K$, *then* $\mu(E) \leq \eta$ *implies that there exists* $z \in Ball(Z)$ *such that* $\|1_{\sim E} Tz\|_p > \kappa$.

Proof. Without loss of generality we may assume that $\mu(E) > 0$. Observe that for any measurable set A one has

$$C_{1,p}(1_A T\colon Z \to L_1(\mu)) \leq \mu(A)^{1/p^*} \|1_A T\colon Z \to L_p(\mu)\|.$$

The lemma follows by using this observation for $A = \sim E$ and $A = E$, because

$$\begin{aligned}\|1_{\sim E} T\colon Z \to L_p(\mu)\| &> C_{1,p}(1_{\sim E} T) \geq C_{1,p}(T) - C_{1,p}(1_E T)\\ &\geq K - \mu(E)^{1/p^*} \|1_E T\colon Z \to L_p(\mu)\| \geq K(1 - \mathcal{C}\eta^{1/p^*}) \geq \kappa.\end{aligned}$$

$\square$

$\square$

proof of 2.2.1. Clearly, we may assume that $\|T\| = 1$. Put $\kappa = \frac{1}{2}K$, $\eta = (2\mathcal{C})^{-p^*}$, $\delta = \left(\frac{2^{1/p}}{\kappa}\right)^{p^*}$ and let us start with $E = \emptyset$. Since

$$\|1_{\sim E}T\colon Z \to L_p(\mu)\| > \kappa,$$

using Lemma 2.2.2 we can define z_1 and F_1 so that $\mu(F_1) < \delta$ and $\|1_{F_1}Tz_1\|_1$ satisfies (1). Suppose now that, for some $i \geq 1$, we have already defined $z_1, \ldots, z_i$ and $F_1, \ldots, F_i$. Let $E = \bigcup_{j \leq i} F_j$.

As long as $\mu(E) \leq \eta$, Lemma 3 guarantees that we can use Lemma 2.2.2 again in order to choose z_{i+1} and F_{i+1} so that $F_{i+1} \cap E = \emptyset$, $\mu(F_{i+1}) < \delta$ and $\|1_{F_{i+1}}Tz_{i+1}\|_1$ satisfies the estimate (1). Therefore this procedure can be applied more than η/δ times. Since we have been assuming $\|T\| = 1$, this yields the promised lower estimate for m and completes the proof of the proposition. □

The next proposition shows that we can actually get a somewhat stronger conclusion to 2.2.1; namely, for some k proportional to m, the identity on l_1^k can be factored through T. Recall that, for a pair of linear operators $T\colon X \to Y$ and $U\colon X_1 \to Y_1$, the factorization constant of U through T is defined to be

$$\gamma_T(U) = \inf\{\|A\|\|B\| : A\colon X_1 \to X,\ B\colon Y \to Y_1,\ U = BTA\}.$$

We let $\gamma_T(U) = \infty$ if no such factorization exists. We also put

$$\gamma_T(Z) = \gamma_T(id_Z\colon Z \to Z).$$

Proposition 2.2.4. *Let $T\colon Z \to L_1(\mu)$ be a bounded linear operator. Suppose that $z_1, \ldots, z_m$ are in the unit ball of Z and $F_1, \ldots, F_m$ are mutually disjoint μ-measurable sets such that, for $i = 1, \ldots, m$,*

$$\|1_{F_i}Tz_i\| \geq \delta\|T\| > 0.$$

Then for some $k \geq \frac{1}{8}\delta m$ there exist linear operators $A\colon l_1^k \to Z$ and $B\colon L_1(\mu) \to l_1^k$ such that $BTA = id_{l_1^k}$ and $\|A\|\|B\|\|T\| \leq 2\delta^{-1}$; i.e., $\gamma_T(l_1^k) \leq 2\delta^{-1}\|T\|^{-1}$ for some $k \geq \frac{1}{8}\delta m$.

For the proof we need two basically known lemmas (see [2]).

Lemma 2.2.5. *Let $x_1, \ldots, x_m$ be elements of $L_1(\mu)$ and let $A_1, \ldots, A_m$ be mutually disjoint μ-measurable sets. If $1 < k \leq \frac{1}{2}m$, then there exists a subset $D \subset \{1, \ldots, m\}$ with $|D| = k$ such that for each $i \in D$ then we have:*

$$\sum_{j \in D \sim \{i\}} \int_{A_j} |x_i|\, d\mu \leq (2k-1)\binom{m}{2}^{-1} \sum_{i=1}^{m} \|x_i\|.$$

Proof. Setting $a_{ij} = \int_{A_j} |x_i|\, d\mu$, for $i, j = 1, \ldots, m$, and $\alpha = \sum_{1 \leq i \neq j \leq m} a_{ij}$, we have

$$\alpha \leq \sum_{i=1}^{m} \|x_i\|.$$

Put $s = 2k$, $\mathcal{E} = \{E \subset \{1, \ldots, m\} : |E| = s\}$. Write, for $E \in \mathcal{E}$,

$$\alpha(E) = \sum_{i \in E} \sum_{j \in E \sim \{i\}} a_{ij}.$$

It is easy to see that $\sum_{E\in\mathcal{E}}\alpha(E)=\binom{m-2}{s-2}\alpha=|\mathcal{E}|\binom{s}{2}\binom{m}{2}^{-1}\alpha$, hence we can pick $E_0\in\mathcal{E}$ so that $\alpha(E_0)\le\binom{s}{2}\binom{m}{2}^{-1}\alpha$. Let $F=\{i\in E_0:\sum_{j\in E_0\sim\{i\}}a_{ij}\ge\frac{1}{k}\alpha(E_0)\}$. Then F has at most k elements and, clearly, each k–element subset $D\subseteq E_0\sim F$ has the required property. □ □

Lemma 2.2.6. *Let $U\colon l_1^k\to L_1(\mu)$ be a linear operator, $\|U\|\le 1$. Suppose there exist μ-measurable sets $F_1,\dots,F_k$ such that for $i=1,\dots,k$*

$$\|1_{F_i}Ue_i\|\ge\delta,\qquad\sum_{1\le i\ne j\le k}\|1_{F_j}Ue_i\|\le\gamma,$$

where $0<\gamma<\delta$. Then there exists a linear operator $Q\colon L_1(\mu)\to l_1^k$ such that $QU=id_{l_1^k}$ and $\|Q\|\le(\delta-\gamma)^{-1}$.

Proof. Define $W\colon L_1(\mu)\to l_1^k$ by the formula $Wf=(\int fg_i\,d\mu)_{i=1}^k$, where

$$g_i=1_{F_i}\,(Ue_i)\text{ for }i=1,\dots,k.$$

It is easy to check that $\|W\|\le 1$ and for $x\in l_1^k$ one has $\|WUx\|\ge(\delta-\gamma)\|x\|$. Therefore the operator $Q=(WU)^{-1}W$ has the required properties. □ □

Proof of Proposition 2.2.4. We may assume that $\|T\|=1$. Apply 2.2.5 with $\eta=\frac{1}{2}\delta$ and $x_i=Tz_i$ for $i=1,\dots,m$. This yields a set $D\subset\{1,\dots,m\}$, with $|D|=k\ge\frac{1}{8}\delta m$, which satisfies the assertion of Lemma 5. Writing $\{z_i:i\in D\}=\{f_1,\dots,f_m\}$ we can define the operator A by the formula $Ae_i=f_i$, for $i=1,\dots,k$. The existence of the operator B follows then immediately from ‘2.2.6. □

We shall combine Propositions 2.2.1 and 2.2.4 in 2.2.8 below. Before doing that we would like to state a dual version of Proposition 4. Note that, if $\dim X_1,\dim Y_1<\infty$, then $\gamma_T(U)=\gamma_{T^*}(U^*)$. This follows from the principle of local reflexivity ([10], p.33). □

Corollary 2.2.7. *Let $V\colon l_\infty^m\to X$ be an operator of norm 1 such that $\|Ve_i\|\ge\delta>0$ for $i=1,\dots,m$. Then $\gamma_V(l_\infty^k)\le 2\delta^{-1}$ for some $k\ge\frac{1}{8}\delta m$.*

Proof. Let $Z=X^*$. Pick norm one elements $z_1,\dots,z_m$ in Z such that $z_i(Ve_i)\ge\delta$ for $i=1,\dots,m$. Using Proposition 2.2.4 we obtain $\gamma_{V^*}(l_1^k)\le 2\delta^{-1}$ for some $k\ge\frac{1}{8}\delta m$. Since $\gamma_V(l_\infty^k)=\gamma_{V^*}(l_1^k)$ this completes the proof. □ □

2.2.3 The L_1 result

The main results of this section are Theorem 2.2.11 and Corollary 2.2.12 below. **Corollary 12.A** states roughly that in any good factorization of a natural embedding of l_2^n into $L_1(0,1)$ (for example, the embedding sending the unit vector basis of l_2^n to the first n Rademacher functions) through an L_1 space, the operator between the two L_1 spaces preserves an l_1^k space with k exponential in n. We begin however with a theorem of a more general nature which is a corollary to Propositions 2.2.1 and 2.2.4 The assumptions in both this theorem and Theorem 2.2.11 are stated in terms of factorization constants of an operator into an L_1 space through L_p spaces via changes of densities. The relation between these constants and factorizations of natural embeddings was one of the main tools in [1]. We shall return to this relation in the proof of Corollary 2.2.12.

Theorem 2.2.8. *Let $T : Z \to L_1(\mu)$ be a linear operator such that $T \neq 0$ and $C_{1,q}(T) < \infty$, where $1 < p < q \leq \infty$. Set $\sigma = 1 - \frac{q^*}{p^*}$, $\quad \Delta = \|T\|^\sigma C_{1,q}(T)^{1-\sigma}/C_{1,p}(T)$.
Then :*

$$\gamma_T(l_1^k) \leq 2(4\Delta)^{\frac{1}{\sigma}}\|T\|^{-1}$$

for some:

$$k \geq \frac{1}{8}(4\Delta)^{\frac{-1}{\sigma}}\left(\frac{C_{1,p}(T)}{8\|t\|}\right)^{P^*}$$

Proof. By Maurey's result [8] quoted in [1], for each $r \in (1, \infty]$ there is a nonnegative function $\phi_r \in L_1(\mu)$ such that $\int \phi_r \, d\mu = 1$ and

$$\|\phi_r^{-1}T \colon Z \to L_r(\phi_r d\mu)\| = C_{1,r}(T)$$

(this uses the convention $\frac{0}{0} = 0$). Set $\phi = \frac{1}{2}(\phi_p + \phi_q)$. Then for $r = q$ and $r = p$

$$\|\phi^{-1}T \colon Z \to L_r(\phi \, d\mu)\| \leq 2^{1/r^*} C_{1,r}(T).$$

Consider the operator $T_1 = \phi^{-1}T \colon Z \to L_1(\phi \, d\mu)$. Since $C_{1,r}(T_1) = C_{1,r}(T)$ for $r \in (1, \infty]$, applying Proposition 2.2.1 to the operator T_1, we have $\mathcal{C} \leq 2^{1/p^*}$. We can estimate for each i

$$\|1_{F_i} T_1 z_i\|_{L_q(\phi \, d\mu)} \leq \|T_1 \colon Z \to L_q(\phi \, d\mu)\| \leq 2^{1/q^*} C_{1,q}(T).$$

Hence we obtain elements $z_1, \ldots, z_m$ in $Ball(Z)$ and sets $F_1, \ldots, F_m$ so that

$$\|1_{F_i} T_1 z_i\|_{L_1(\phi \, d\mu)}^\sigma \geq 2^{-\frac{1}{p}-1}\big(2^{\frac{1}{q^*}} C_{1,q}(T)\big)^{\sigma-1} C_{1,p}(T) = 2^{-2}\Delta^{-1}\|T\|^\sigma$$

and $m > \big(\frac{C_{1,p}(T)}{8\|T\|}\big)^{p^*}$. Therefore, we have

$$\|1_{F_i} T z_i\|_1 = \|1_{F_i} T_1 z_i\|_{L_1(\phi \, d\mu)} \geq (4\Delta)^{-\frac{1}{\sigma}}\|T\|.$$

Now we simply apply Proposition 2.2.4 to T, $z_1, \ldots, z_m$ and $F_1, \ldots, F_m$. □ □

We next state two corollaries to 2.2.8 concerning the dual situation.

Corollary 2.2.9. **??** *Let $U \colon C(K) \to X$ be a linear operator such that $0 < \pi_r(U) < \infty$, where $1 \leq r < t < \infty$. Set $\sigma = 1 - \frac{r}{t}$, $\Delta = \|U\|^\sigma \pi_r(U)^{1-\sigma}/\pi_t(U)$.
Then*

$$\gamma_U(l_\infty^k) \leq 2(4\Delta)^{\frac{1}{\sigma}}\|U\|^{-1} \quad \text{for some} \quad k \geq \frac{1}{8}(4\Delta)^{-1/\sigma}\Big(\frac{\pi_t(U)}{8\|U\|}\Big)^t.$$

Proof. This follows easily from 2.2.8, because $C(K)^*$ is an L_1 space, $\gamma_{U^*}(l_1^k) = \gamma_U(l_\infty^k)$ and $C_{1,p}(U^*) = \pi_{p^*}(U)$ for $1 < p \leq \infty$ (see [3]). □

□

Corollary 2.2.10. *If $U \colon l_\infty^N \to X$, $t > 1$ and $\pi_t(U) = c\|U\| > 0$, then $\gamma_U(l_\infty^k) \leq 2(\frac{4}{c})^{t^*} N^{t^*-1}\|U\|^{-1}$ for some $k \geq 2^{t^*-3}(\frac{c}{8})^{t^*+t} N^{1-t^*}$.*

This follows by using with $r = 1$, because $\pi_1(U) \leq N\,\|U\|$. □

The next theorem and Corollary 2.2.12 below are the main results of this section.

Theorem 2.2.11. *Let* $T\colon Z \to L_1(\mu)$ *be a bounded linear operator and let* $1 < p < \infty$. *Let* $Z_0 \subseteq Z$. *Suppose that* $n = \dim TZ_0 < \infty$ *and that*

$$C_{1,p}(u) \geq c\|T\| > 0,$$

for each finite rank operator $u\colon Z \to L_1(\mu)$ *such that* $u|_{Z_0} = T|_{Z_0}$.
If $c \geq 2^5$ *and* $\delta = (p-1)n$, *then* $\gamma_T(l_1^k) < 5^\delta$ *for some* $k > 5^{-\delta}\left(2^{1/\delta}\right)^n$.

Corollary 2.2.12. *Let* X *be an* n*-dimensional subspace of* L_1 *for which* $C_{1,p^*}(X) \leq C\sqrt{p^*}$ *for all* $2 \leq p^* < \infty$. *If* j *is the inclusion map from* X *into* L_1 *and* $j = UW$ *with* $W\colon X \to Z$, $U\colon Z \to L_1$, $\|W\| \leq 1$ *and* Z *is an* L_1 *space, then* $\gamma_U(l_1^k) \leq 5^{2D}$ *for some* $k \geq 5^{-2D}\, 2^{n/(2D)}$, *where* $D = (2^8 C\|U\|)^2$.

Proof. The proof of the corollary is very similar to the proof of **Corollary 1.5** in [1]; there are, however, some changes. Here is an outline of the proof: Let $Z_0 = WX$ and define p by $p^* = \frac{n}{D}$. If $n < 2D$ the conclusion is obvious, so we may assume that $n \geq 2D$. Then $\delta = (p-1)n$ satisfies $\delta \leq 2D$, so, by Theorem 2.2.11, it is enough to prove that for any extension $V\colon Z \to L_1(\mu)$ of $U|_{Z_0}$, one has $C_{1,p}(V) \geq 2^5\|U\|$.

Now, for any extension $\tilde{U}\colon Z \to X$ of $U|_{Z_0}$, we have $id_X = \tilde{U}W$ and hence, using a weak form of Grothendieck's inequality, we obtain

$$n^{1/2} = \pi_2(id_X) \leq \|W\|\pi_2(\tilde{U}) \leq 2\,\gamma_2(\tilde{U}).$$

We can now apply **Theorem 1.3** in [1] with $\beta = \frac{1}{2}n^{1/2}$ to get that for any extension $V\colon Z \to L_1$ of $U|_{Z_0}$

$$C_{1,p}(V) \geq 2^{-3} C_{1,p^*}(X)^{-1} n^{1/2}.$$

With the choice $p^* = n/D$ we get the desired estimate

$$C_{1,p}(V) \geq 2^{-3} C^{-1} D^{1/2} = 2^5\|U\|. \qquad \square$$

□

Corollary 12.B strengthens **Corollary 12.A** in the same way that **Theorem 5.1** in [1] strengthens **Corollary 1.5** in [1].

Corollary 2.2.13. *Suppose that* $X \subset L_1$, $\dim X = n$, *and* $C_{1,p^*}(X) \leq C\sqrt{p^*}$ *for all* $2 \leq p^* < \infty$. *Let* Y *be a Banach space whose dual has finite cotype* q *constant* $C_q(Y^*)$ *and let* $Q\colon Y \to L_1$ *be an operator for which*

$$Q(Ball(Y)) \supset Ball(X).$$

Let $Q = UW$ *be any factorization of* Q *through an* L_1 *space with* $\|W\| \leq 1$. *Then for some absolute constant* η, $\gamma_U(l_1^k) \leq 5^D$ *for some* $k \geq 5^{-D}\, 2^{n/D}$, *where*

$$D = \eta C^2 q C_q^2(Y^*)\|U\|^2.$$

Sketch of proof. Follow the proof of 2.2.8 in [1]. (with r replaced by p) up to the place on p. 98 where it is proved that $C_{1,p}(U) \geq 2$. Of course, now we need and can assure that $C_{1,p}(\tilde{U}) \geq 2^5\|U\|$ for any extension $\tilde{U}\colon Z \to L_1$ of the restriction of U to $U^{-1}(X)$. Then apply Theorem 2.2.11. □ □

To prove Theorem 2.2.11 we need to introduce some notation and some preliminary results. Given two Banach spaces Z and W the space of all bounded linear operators between them is denoted by $B(Z,W)$, while $F(Z,W)$ is the set of those $u \in B(Z,W)$ such that $u < \infty$. By α we denote a norm on $F(Z,W)$ such that $\alpha(u) \leq \|u\|$ if $u = 1$.

Given a Banach space W and numbers $n, \beta \geq 1$, let us denote by $q_W(n,\beta)$ the least number k such that, whenever $v\colon W \to E$ is a continuous linear operator with $v \leq n$, there exists $P \in F(W,W)$ such that $vP = v$, $\|P\| \leq \beta$ and $P \leq k$. (Of course, we let $q_W(n,\beta) = \infty$, if no such k exists.)

Proposition 2.2.14. *Let $T \in B(Z,W)$ and let $Z_0 \subseteq Z$, $\dim T(Z_0) = n < \infty$. If $1 \leq \beta < \infty$ and $q_W(n,\beta) < \infty$, then there exists $P \in F(W,W)$ such that $\|P\| \leq \beta$, $P \leq q_W(n,\beta)$ and*

$$\alpha(PT) \geq \inf\{\alpha(u) : u \in F(Z,W), u|_{Z_0} = T|_{Z_0}\}.$$

Proof. Write $Y = \{u \in F(Z,W) : u|_{Z_0} = T|_{Z_0}\}$ and $A = \inf\{\alpha(u) : u \in Y\}$. By the Hahn–Banach theorem there is a norm one functional Φ on $(F(Z,W),\alpha)$ such that $\Phi(u) = A$ for each $u \in Y$. Observe that if $S\colon W \to Z^{**}$ is the linear operator defined by

$$(Sw)\,(z^*) = \Phi(z^* \otimes w),$$

then for all $u \in F(Z,W)$ one has

$$\Phi(u) = Tr(Su) = Tr(u^{**}S).$$

Clearly, our assumption on α yields $\|S\| \leq 1$. Moreover, for all $u \in Y$ one has

$$(T-u)^{**}S = 0. \tag{2}$$

Indeed, since $\operatorname{Ker}((T-u)^{**}) \supseteq (\operatorname{Ker}(T-u))^{\perp\perp} \supseteq {Z_0}^{\perp\perp}$, it suffices to verify that $SW \subseteq Z_0^{\perp\perp}$. The latter inclusion is obvious, because if $w \in W$, $z_0^* \in Z_0^{\perp}$ then we have $(Sw)(z_0^*) = \Phi(z_0^* \otimes w) = 0$, since z_0^* annihilates Z_0. Since $u_0 = \dim TZ_0 = n$, for some $u_0 \in Y$, and since (2) implies that $T^{**}S = u^{**}S$ for each $u \in Y$, we obtain that $T^{**}S \leq n$. Hence, by the definition of $q_W(n,\beta)$, there is a $P \in F(W,W)$ such that $T^{**}SP = T^{**}S$, $\|P\| \leq \beta$ and $P \leq q_W(n,\beta)$. Observe that, if u is any element of Y, then

$$\alpha(PT) \geq \Phi(PT) = Tr(SPT) = Tr(T^{**}SP) = Tr(T^{**}S) = Tr(u^{**}S) = \Phi(u) = A. \qquad \square$$

$\square$

Lemma 2.2.15. *If $W = L_1(\mu)$ and $0 < \epsilon < 1$, then $q_W(n,(1-\epsilon)^{-1}) < (\frac{2}{\epsilon}+1)^n$.*

Proof. Let $u\colon W \to E$ have n. Write $u = UQ_0$, where $Q_0\colon W \to W/(\operatorname{Ker}\ u)$ is the quotient map and let $F = W/(\operatorname{Ker}\ u)$. Set $\beta = (1-\epsilon)^{-1}$. Suppose first that for some k there exists an operator $Q\colon l_1^k \to F$ such that $\|Q\| \leq \beta$ and $Q(Ball(l_1^k)) \supseteq Ball(F)$. By the lifting property of l_1^k there is $Q_1\colon l_1^k \to W$ such that $\|Q_1\| \leq \|Q\| \leq \beta$ and $Q = Q_0Q_1$. By the lifting property of $W = L_1(\mu)$ there is $Q_2\colon W \to l_1^k$ such that $\|Q_2\| \leq \|Q_0\| \leq 1$ and $Q_0 = QQ_2$. Let $P = Q_1Q_2$. Then $\|P\| \leq \beta$, $P \leq k$ and

$$u = UQ_0 = UQQ_2 = UQ_0Q_1Q_2 = uP.$$

Now the well–known volume argument shows that the unit sphere of F contains an ϵ–net (where $(1-\epsilon)^{-1} = \beta$) of cardinality $k < \frac{1}{2}(\frac{2}{\epsilon}+1)^n$. Using this fact one easily constructs the operator $Q\colon l_1^k \to F$ with the two properties which we have used above. $\square$ $\square$

Theorem 2.2.11 can now be obtained by letting $\epsilon = \frac{1}{2}$ and replacing c by 2^5 in the following proposition.

Proposition 2.2.16. *Let $u\colon W \to E$ have n. Write $u = UQ_0$, where $Q_0\colon W \to W/(\mathrm{Ker}\ u)$ is the quotient map and let $F = W/(\mathrm{Ker}\ u)$. Set $\beta = (1-\epsilon)^{-1}$. Suppose first that for some k there exists an operator $Q\colon l_1^k \to F$ such that $\|Q\| \le \beta$ and $Q(Ball(l_1^k)) \supseteq Ball(F)$. By the lifting property of l_1^k there is $Q_1\colon l_1^k \to W$ such that $\|Q_1\| \le \|Q\| \le \beta$ and $Q = Q_0Q_1$. By the lifting property of $W = L_1(\mu)$ there is $Q_2\colon W \to l_1^k$ such that $\|Q_2\| \le \|Q_0\| \le 1$ and $Q_0 = QQ_2$. Let $P = Q_1Q_2$. Then $\|P\| \le \beta$, $P \le k$ and*

$$u = UQ_0 = UQQ_2 = UQ_0Q_1Q_2 = uP.$$

Now the well–known volume argument shows that the unit sphere of F contains an ϵ–net (where $(1-\epsilon)^{-1} = \beta$) of cardinality $k < \frac{1}{2}(\frac{2}{\epsilon}+1)^n$. Using this fact one easily constructs the operator $Q\colon l_1^k \to F$ with the two properties which we have used above. □

Proof. Let $\beta = (1-\epsilon)^{-1}$, $W = L_1(\mu)$, $\alpha = C_{1,p}$. Thanks to Lemma 14, we can apply Proposition 13 which yields an operator P on $L_1(\mu)$ such that $\|P\| < \beta$, $P \le N = \frac{1}{2}(\frac{2}{\epsilon}+1)^n$ and $C_{1,p}(PT) \ge c\|T\|$. Clearly,

$$C_{1,\infty}(PT) \le (PT)\|PT\| \le N\|PT\|.$$

Let $q = \infty$. We can now estimate $\gamma_{PT}(l_1^k)$, using Theorem 2.2.8. The resulting inequality, combined with the obvious relation $\gamma_T(l_1^k) \le \|P\|\gamma_{PT}(l_1^k)$, yields the desired estimates for $\gamma_T(l_1^k)$ and k. □

□

2.2.4 The $C(K)$ result

The main result of this section is Theorem 16 and in particular its Corollary 20 which gives a local version of a result of $Peczy'nski$ by showing that an operator from a $C(K)$ space which preserves a copy of l_2^n also preserves a copy of l_∞^k with k an exponent of n.

Theorem 2.2.17. *Let $U\colon C(K) \to X$ be a bounded linear operator and let $1 < t < \infty$. Suppose that $E \subseteq C(K)$, $\dim E = n < \infty$ and let*

$$\pi_t(U|_E) = c\|U\| > 0.$$

If $c \ge 2^5$ and $\alpha = \frac{n}{t-1}$, then $\gamma_U(l_\infty^k) < 5^\alpha\|U\|^{-1}$ for some $k > 5^{-\alpha}\big(2^{1/\alpha}\big)^n$.

For the proof we need a dual version of Lemma 2.2.15 and a proposition. The lemma is a simplified version of a result which was proved in [1].

Lemma 2.2.18. *Let F be a subspace of $C(K)$,* dim $F = n$. *Let $0 < \epsilon < 1$ and $N = \frac{1}{2}(\frac{2}{\epsilon}+1)^n$. Then there is $Q\colon C(K) \to C(K)$ such that $Qf = f$ for $f \in F$, $\|Q\| < (1-\epsilon)^{-1}$ and $Q \le N$.*

Proof. There exists $k \le N$ and an operator $J\colon F \to l_\infty^k$ such that $\|J\| < (1-\epsilon)^{-1}$ and $\|Jf\| \ge \|f\|$ for $f \in F$. By the extension property of $C(K)$, there is $J_1\colon l_\infty^k \to C(K)$ such that $\|J_1\| < (\|J\|(1-\epsilon))^{-1}$ and $J_1(Jf) = f$ for $f \in F$. We let $Q = J_1J_2$, where $J_2\colon C(K) \to l_\infty^k$ is a linear extension of J with $\|J_2\| = \|J\|$. □ □

Theorem 2.2.17 follows easily from the next proposition by letting $\epsilon = \frac{1}{2}$ and replacing c by 2^5.

Proposition 2.2.19. *Let $U\colon C(K) \to X$ satisfy the assumptions of Theorem 16, and let $0 < \epsilon < 1$. Then*

$$\gamma_U(l_\infty^k) < 4\Big(\frac{2}{(1-\epsilon)c}\Big)^{t^*}\Big(\frac{2}{\epsilon}+1\Big)^{n/(t-1)}\|U\|^{-1}$$

for some $k > 4^{t^*-2}\big(\frac{(1-\epsilon)c}{8}\big)^{tt^*}\big(\frac{2}{\epsilon}+1\big)^{-n/(t-1)}$.

Proof. there exists $P\colon C(K) \to C(K)$ such that $\|P\| < (1-\epsilon)^{-1}$, $P < N = \frac{1}{2}\big(\frac{2}{\epsilon}+1\big)^n$ and $Pe = e$ for $e \in E$. It follows that

$$C_{1,t^*}(P^*U^*) \geq \pi_t(UP) \geq \pi_t(UP|_E) = c\|U\|.$$

Since also $C_{1,\infty}(P^*U^*) \leq (UP)\|(UP)^*\| \leq N\|UP\|$, we can finish the proof by applying an argument similar to that in the proof of Proposition 2.2.16 and dualizing. □

□

Corollary 2.2.20. *Let X be a Banach space. Suppose that $t, n > 1$ and there is an operator $U\colon C(K) \to X$ and a subspace $E \subseteq C(K)$, $\dim E = n < \infty$ such that $\pi_t(U|_E) = c\|U\| > 0$, where $c \geq 2^5$.*
Write $\alpha = \frac{n}{t-1}$, $c_1 = (\log 2)(\log \frac{4}{3})/\log 5$. Then, for all $j \leq \min\{5^{-\alpha}\big(2^{1/\alpha}\big)^n, \frac{3}{4}\exp(\frac{c_1 n}{\alpha^2})\}$, X contains a subspace X_j such that $d(X_j, l_\infty^j) < 2$.

Proof. Since $c \geq 32$ using Theorem 2.2.17 we obtain that, if $\alpha = \frac{n}{t-1}$, then

$$\gamma_U(l_\infty^k) < 5^\alpha\|U\|^{-1}$$

for some $k > 5^{-\alpha}\big(2^{1/\alpha}\big)^n$. Consequently, we obtain the inequality

$$\gamma_{(id)_x}(l_\infty^k) \leq \|U\|\gamma_U(l_\infty^k) < 5^\alpha$$

from which we shall deduce a lower estimate for the number

$$j_0 = \min\{m : \gamma_{(id)_x}(l_\infty^m) \geq 2\}$$

Put for brevity $g_i(X) = \gamma_{(id)_x}(l_\infty^i)$

We shall employ the estimate

$$g_{ij}(X) \geq g_i(X)\frac{2}{1+g_j(X)^{-1}}\,, \tag{3}$$

for $i, j = 1, 2, \ldots$, which is the quantitative statement of results of James [[4]] and Giesy [[5]] (see, e.g., [[6]]).

Suppose that $j_0 \leq k$ and let m be an integer such that $j_0^{m+1} > k \geq j_0^m$. Write $A = g_{j_0}(X) \geq 2$, $B = 2/(1+A^{-1})$. Using repeatedly (3), one obtains

$$g_k(X) \geq g_{j_0}^m(X) \geq AB^{m-1} \geq \frac{A}{B^2}B^{\frac{\log k}{\log j_0}}.$$

Since $A \geq 2$, we have $\frac{A}{B^2} \geq \frac{9}{8}$, $B \geq \frac{4}{3}$. Taking logarithms of both sides we obtain the estimate

$$\alpha \log 5 \geq \log(g_k(X)) > (\log \frac{4}{3}) \frac{\log k}{\log j_0}.$$

Using now the estimate $k > 5^{-\alpha}\big(2^{1/\alpha}\big)^n$, we get easily

$$\log j_0 > (\alpha \log 5)^{-1} (\log \frac{4}{3})(\frac{n}{\alpha} \log 2 - \alpha \log 5) = (\log \frac{4}{3})(\frac{\log 2}{\log 5}\frac{n}{\alpha^2} - 1) =: \log M.$$

The latter estimate implies that if $1 < j \leq \min\{k, M\}$ then $g_j(X) < 2$. This implies that X contains a subspace X_j with $d(X_j, l_\infty^j) < 2$ and completes the proof. □

□

Corollary 2.2.21. *Let $U\colon C(K) \to X$ be a linear operator of norm 1. Suppose that for some subspace $E \subseteq C(K)$ such that $d(E, l_2^n) = a$, $n \geq 2$, one has $\|Ux\| \geq b\|x\|$ for $x \in E$, where $b > 0$. Then U is bounded from below by $A_1^{-(a/b)^2}$ on a subspace $G \subseteq C(K)$ such that $d(G, l_\infty^j) < 2$ and $j \geq A_2^{(b/a)^4 n}$, where A_1, A_2 are absolute constants > 1.*

Proof. Since $\pi_t(l_2^n) \geq \sqrt{\frac{n}{t}}$ for $t \geq 1$ [11], we can estimate

$$\pi_t(U|_E) \geq \frac{b}{a}\pi_t(id_{l_2^n}) \geq \frac{b}{a}\sqrt{\frac{n}{t}}\|U\|.$$

Assume first that $n \geq (2^5 a/b)^4$. Letting $t = (2^{-5}b/a)^2 n$ we obtain the estimate $\pi_t(U|_E) \geq 2^5\|U\|$. Let $\alpha = \frac{n}{t-1}$. Using Theorem 16 we obtain, for some $k > 5^{-\alpha}2^{n/\alpha}$, a pair of operators $A\colon l_\infty^k \to C(K)$ and $B\colon X \to l_\infty^k$ such that $\|A\|\|B\| < 5^\alpha$ and $BUA = id_{l_\infty^k}$. Let $F = A(l_\infty^k)$. Clearly, one has $\|Ux\| \geq 5^{-\alpha}\|x\|$ for $x \in F$, and $d(F, l_\infty^k) < 5^\alpha$. Now, if $d(F, l_\infty^k) \geq 2$, then the argument used in the proof of Corollary 19 can be applied to F. Since $g_k(F) < 5^\alpha$ this will produce a subspace $G \subseteq F$ such that $j = \dim G > \frac{3}{4}\exp(c_1 \frac{n}{\alpha^2}) - 1$ and $d(G, l_\infty^k) < 2$. This yields the following conditions on numbers A_1, A_2

$$5^{n/(t-1)} \leq A_1^{(a/b)^2}, \qquad j \geq A_2^{(b/a)^4 n}.$$

If $n < (2^5 a/b)^4$, then we let G be any 2–dimensional subspace of E, so that $d(G, l_\infty^2) < 2$. This gives the following conditions on numbers A_1, A_2

$$b \geq A_1^{-(a/b)^2}, \qquad 2 \geq A_2^{(2^5)^4}.$$

It is not difficult to check that one can find $A_1, A_2 > 1$ which satisfy all the above conditions. □ □

2.2.5 A space with very non-unconditional structure

The main result here is Theorem 2.2.22 which, roughly speaking, shows the existence of an m-dimensional space G which is contained in an n-dimensional space Z with an unconditional basis only if n is an exponent of m. Moreover any such Z must contain l_∞^k with k an exponent of m. This solves part of **problem 11.4(b)** in [12].

Recall that the gl norm of a linear operator $T\colon X \to Y$ is defined by the formula

$$gl(T) = \sup\{\gamma_1(UT) : U\colon Y \to l_2, \quad \pi_1(U) \leq 1\},$$

and that one writes $gl(X) = gl(id_X)$. Recall [13] also that the unconditional constant of X is greater than or equal to $gl(X)$.

Theorem 2.2.22. *There is $\delta > 0$ such that for each $m \geq 2$ there is a Banach space G_m,* $\dim G_m = m$*, with the following property. If Z is a Banach space which contains an isometric copy of G_m, then there is a subspace Z_1 of Z such that $d(Z_1, l_\infty^k) < 2$, where $k \geq$* $\exp(\delta m\, gl(Z)^{-4})$*.*

In fact, a somewhat stronger version of this theorem follows by applying Lemma 2.2.24 to the space obtained in Lemma 2.2.23. A stronger version of Lemma 2.2.23 appears as Th. 7.1 in [[7]].

Lemma 2.2.23. *There is a constant $B < \infty$ such that for $n = 1, 2, \dots$ there is a Banach space F_n,* dim *$F_n = 2n$, and a linear operator $v\colon l_2^n \to F_n$ such that $\|ve\| \geq \|e\|$ for $e \in l_2^n$ and $\pi_1(v^*) \leq B$.*

Proof. Consider a linear isometry $U\colon L_2^{3n} \to L_2^{3n}$. Write

$$E_1 = U([e_1, \dots, e_{2n}]), \qquad E_2 = U([e_{n+1}, \dots, e_{3n}]).$$

It is well–known (see e.g. [14], [7], or [15] Cor.7.4) that for "most choices" of U one has

$$\|f\|_{L_2^{3n}} \leq b\|f\|_{L_1^{3n}},$$

for $f \in E_1 \cup E_2$, where b can be taken independent of n. Let us fix a pair E_1, E_2 with the latter property. Put $F_n = L_1^{3n}/E_2^\perp$ and let

$$u\colon E_1/E_2^\perp \to L_1^{3n}/E_2^\perp = F_n$$

be the natural map. If $E_1/E_2^\perp$ is given the norm induced from $L_2^{3n}/E_2^\perp$, then $E_1/E_2^\perp$ is isometric to l_2^n and our choice of E_1 yields the estimate $\|e\| \leq b\|ue\|$ for $e \in E_1/E_2^\perp$. Now u^* can be regarded as the composition of the embedding map $i_{\infty,2}^{E_2}\colon (E_2)_\infty \to (E_2)_2$ with the orthogonal projection P from L_2^{3n} onto $E_1 \cap E_2$. Hence our choice of E_2 yields

$$\pi_1(u^*) \leq \|P\|\pi_1(i_{\infty,2}^{E_2}) \leq \pi_1(i_{\infty,1}^{E_2})\|i_{1,2}^{E_2}\| \leq b.$$

This shows that the operator $v = bu$ has the required properties, if $B = b^2$. □ □

Lemma 2.2.24. *Let F be a Banach space and let $v\colon l_2^n \to F$, $1 < t < \infty$. Suppose that*

$$\pi_t(v) \geq c\|v\| > 0, \qquad \pi_1(v^*) \leq B\|v\|.$$

Let $j\colon F \to Z$ be a linear operator such that $\|jf\| \geq \|f\|$ for $f \in v(l_2^n)$ If $c \geq 2^5 B\, gl(j)$, then $Z \supset Z_1$ such that $d(Z_1, l_\infty^k) < 2$ and $k =$ $\dim Z_1 \geq \min\{5^{-\alpha}\big(2^{1/\alpha}\big)^n, \frac{3}{4}\exp(\frac{c_1 n}{\alpha^2})\}$*, where α and c_1*

Proof. Observe that, since $gl(j^*) = gl(j)$, one has

$$\gamma_\infty(ju) = \gamma_1((ju)^*) \leq \pi_1(v^*)gl(j^*) \leq B.gl(j)\|v\|.$$

Consider a $C(K)$–factorization of $jv\colon l_2^n \to Z^{**}$, say $jv = Ui$, where

$$\|i\colon l_2^n \to C(K)\| = 1, \qquad \|U\colon C(K) \to Z^{**}\| \leq B\, gl(j)\|v\|.$$

' Put $E = i(l_2^n)$, then

$$\pi_t(U|_E) \geq \pi_t(Ui) \geq \pi_t(v) \geq c\|v\| \geq 2^5\|U\|.$$

Since

$$\gamma_{(id)_{z^{**}}}(l_\infty^k) = \gamma_{(id)_z}(l_\infty^k)$$

□

Proof of Theorem 2.2.22. We may assume that $m > 2(2^5 B\, gl(Z))^4$, where B is the constant from Lemma 2.2.23. (If not, we just let $\delta = \frac{1}{4}(2^5 B)^{-4}$ and G_m can be arbitrary space of dimension m.) Let $G_{2n} = F_n$ and $G_{2n+1} = F_n \oplus l_1^1$ for $n \geq (2^5 B)^4$. Fix an m and let Z be a Banach space and $j\colon G_m \to Z$ an isometric embedding, so that $gl(j) \leq gl(Z)$. Let $v\colon l_2^n \to G_m$ be the operator obtained from that in Lemma 22 (here $m = 2n$ or $m = 2n+1$). Observe that, for $t \geq 1$,

$$\pi_t(v) \geq \pi_t(id_{l_2^n}) \geq \sqrt{\frac{n}{t}}.$$

Letting $t = (2^5 B\, gl(j))^{-2} n$ and applying Lemma 2.2.24, one can easily find the absolute constant δ needed in 2.2.22. □ □

Remark It follows from ([[7]], Th. 7.1) that the spaces G_n in Theorem2.2.22
can be chosen to have uniform cotype 2 constants.

2.3 Convex Bodies with Few Faces

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Abstract
It is proved that if $u_1, \ldots, u_n$ are vectors in $\mathbb{R}^k, k \leq n, 1 \leq p < \infty$ and

$$r = \left(\frac{1}{k} \sum_1^n |u_i|^p \right)^{\frac{1}{p}}$$

then the volume of the symmetric convex body whose boundary functionals are $\pm u_1, \ldots, \pm u_n$, is bounded from below as

$$|\{x \in \mathbb{R}^k \colon \ |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{\frac{1}{k}} \geq \frac{1}{\sqrt{\rho} r}.$$

An application to number theory is stated.

A.M.S. (1980) Subject Classification: 52A20, 10E05

(1)Partially supported by N.S.F. DMS-8807243.

2.3.1 Introduction

In [16], Vaaler proved that if $Q_n = [-\frac{1}{2}, \frac{1}{2}]^n$ is the central unit cube in $\mathbb{R}^n$ and U is a subspace of $\mathbb{R}^n$ then the volume $|U \cap Q_n|$, of the section of Q_n by U is at least 1. This result may be reformulated as follows: if $u_1, \ldots, u_n$ are vectors in $\mathbb{R}^k, 1 \leq k \leq n$ whose Euclidean lengths satisfy $\sum\limits_1^n |u_i|^2 \leq k$ then

$$|\{x \in \mathbb{R}^k \colon \ |\langle x, u_i \rangle| \leq 1 \text{ for every } i\}|^{\frac{1}{k}} \geq 2.$$

A related theorem, (Theorem 2.3.1, below) in which the condition $\sum |u_i|^2 \leq k$ is replaced by $\max\limits_i |u_i| \leq 1$ was proved by Carl and Pajor [C-P] and Gluskin [G]. Gluskin's methods enable him to obtain sharp results in limiting cases which in turn have applications in harmonic analysis. Results closely related to Theorem 1 were also obtained by Bárány and Füredi [17] and Bourgain, Lindenstrauss and Milman [18].

Theorem 2.3.1. *There is a constant $\delta > 0$ so that if $u_1, \ldots, u_n \in \mathbb{R}^k, 1 \leq k \leq n$ are vectors of length at most 1 then*

$$|\{x \in \mathbb{R}^k \colon \ |\langle x, u_i \rangle| \leq 1 \textit{ for every } i\}|^{\frac{1}{k}} \geq \frac{\delta}{\sqrt{1 + \log \frac{n}{k}}}.$$

The estimate is best possible if n is at most exponential in k, apart from the value of the constant δ. This is demonstrated by an example which had appeared some time earlier in a paper of Figiel and Johnson, [F-J]. Theorem 1 gives a lower bound on the volume ratios of the unit balls of k-dimensional subspaces of ℓ_∞^n and hence on the distance of these subspaces from Euclidean space.

Regarding Theorem 2.3.1 as a "$p = \infty$" version of Vaaler's "$p = 2$" result, Kashin asked whether a similar result holds for $2 < p < \infty$. This question is answered in the affirmative by the following theorem.

Theorem 2.3.2. *Suppose* $u_1, \ldots, u_n \in \mathbb{R}^k$ *with* $k \le n, 1 \le p < \infty$ *and let* $r = (\frac{1}{k}\sum\limits_1^n |u_i|^p)^{\frac{1}{p}}$. *Then*

$$|\{x \in \mathbb{R}^k \colon\ |\langle x, u_i\rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \ge \begin{cases} \frac{2\sqrt{2}}{\sqrt{p}r} & \textit{if} \quad p \ge 2 \\ \frac{1}{r} & \textit{if} \quad 1 \le p \le 2 \end{cases}$$

The lower bound is best possible (up to a constant) provided $e^p k \le n \le e^k$.

Remark. The slightly stronger result for $p \ge 2$ is isolated since for $p = 2$ it gives back exactly Vaaler's result.

Theorem 2.3.1 follows immediately from Theorem 2.3.2 by a standard optimisation argument. If $(u_i)_1^n$ in $\mathbb{R}^k$ all have norm at most 1 then for any $p \in [1, \infty)$,

$$\left(\frac{1}{k}\sum_1^n |u_i|^p\right)^{\frac{1}{p}} \le \left(\frac{n}{k}\right)^{\frac{1}{p}}$$

so that

$$|\{x\colon\ |\langle x, u_i\rangle| \le 1 \text{ for every } i\}|^{\frac{1}{k}} \ge \frac{2\sqrt{2}}{\sqrt{p}(\frac{n}{k})^{\frac{1}{p}}}$$

(for $p \ge 2$) and the latter is at least $\frac{2}{\sqrt{e}\sqrt{1+\log\frac{n}{k}}}$ when $p = 2(1 + \log\frac{n}{k})$.

With the careful use of well-known methods for estimating the entropy of convex bodies it is possible to obtain more general (but less precise) estimates than that provided by Theorem 2; (see [B-P]). The purpose of this paper is to provide a very short proof of Theorem 2 and, a fortiori, Theorem 1.

Vaaler originally proved his theorem because of its applications to the geometry of numbers. The last section of this paper includes a statement of the generalisation of Siegel's lemma which follows from Theorem 2.

2.3.2 The lower bound

The proof of Theorem 2.3.2 makes use of the following result from [20] which was designed to extend Vaaler's theorem in a different direction: it estimates the volumes of sections of the unit balls of the spaces $\ell_p^n, 1 \le p \le \infty$. For $1 \le p \le \infty, n \in \mathbb{N}$ let

$$B_p^n = \left\{ x \in \mathbb{R}^n \colon \ \sum_1^n |x_i|^p \leq 1 \right\}$$

be the unit ball of ℓ_p^n.

Theorem 2.3.3. *Let U be a k-dimensional subspace of $\mathbb{R}^n$; if $1 \leq p \leq q \leq \infty$ then*

$$\frac{|B_p^n \cap U|}{|B_p^k|} \leq \frac{|B_q^n \cap U|}{|B_q^k|}. \qquad \square$$

Remark 1. *The case $p = 2, q = \infty$ is Vaaler's theorem since then, the left side is 1 and the inequality states that*

$$|B_\infty^n \cap U| \geq |B_\infty^k| \geq 2^k.$$

For notational convenience, the proof of Theorem 2.3.2 is divided into several short lemmas. The first is no more than a convenient form of Hölder's inequality. For $k \in \mathbb{N}, S^{k-1}$ will denote the Euclidean sphere in $\mathbb{R}^k$ and $\sigma = \sigma_{k-1}$, the rotationally invariant probability measure on S^{k-1}. Also let v_k be the volume of the Euclidean unit ball in $\mathbb{R}^k$.

Lemma 2.3.4. *Let C and B be symmetric convex bodies in $\mathbb{R}^k$ with Minkowski gauges $\|\cdot\|_C$ and $\|\cdot\|_B$ respectively. Then for $p > 0$*

$$\left(\frac{|C|}{|B|}\right)^{\frac{1}{k}} \geq \left(\frac{k+p}{k|B|}\int_B \|x\|_C^p dx\right)^{-\frac{1}{p}}.$$

Proof.

$$\left(\frac{|C|}{|B|}\right)^{\frac{1}{k}} = \left(\frac{v_k}{|B|}\int_{s^{k-1}}\left(\frac{\|\theta\|_B}{\|\theta\|_C}\right)^k \int_0^{\|\theta\|_B^{-1}} r^{k-1} dr d\sigma(\theta)\right)^{\frac{1}{k}} = \left(\frac{1}{|B|}\int_B \left(\frac{\|x\|_B}{\|x\|_C}\right)^k dx\right)$$

(Lemma 2.3.4 appears in [20]as Corollary 2.2.) ☐

Lemma 2.3.5. *Suppose $u_1, \ldots, u_n \in \mathbb{R}^k$ with $k \leq n$ and $1 \leq p < \infty$. Then*

$$|\{x \in \mathbb{R}^k : |\langle x_i, u_i\rangle| \leq 1 \ for\ every\ i\}|^{\frac{1}{k}} \geq 2\left(\frac{k+p}{k}\sum_{i=1}^n \frac{1}{|B_p^k|}\int_{B_p^k} |\langle x, u_i\rangle|^p dx\right)^{\frac{-1}{p}}$$

Proof. Define $T \colon \mathbb{R}^k \to \mathbb{R}^n$ by $(Tx)_i = \langle x, u_i\rangle, 1 \leq i \leq n$ and let $U = T(\mathbb{R}^k)$. The problem is to estimate from below

$$|T^{-1}(B_\infty^n)|^{\frac{1}{k}} = |T^{-1}(U \cap B_\infty^n)|^{\frac{1}{k}}.$$

☐

By Theorem 2.3.3,

$$|U\bigcap B_\infty^n|^{\frac{1}{k}} \geq 2\left(\frac{|U\bigcap B_p^n|}{|B_p^k|}\right)^{\frac{1}{k}} \text{ and so } |T^{-1}(B_\infty^n)|^{\frac{1}{k}} \geq 2\left(\frac{|T^{-1}(B_p^n)|}{|B_p^k|}\right).$$

Regard T as an operator: $\ell_p^k \to \ell_p^n$. Then by Lemma 2.3.4,

$$2\left(\frac{|T^{-1}(B_p^n)|}{|B_p^k|}\right) \geq 2\left(\frac{k+p}{k|B_p^k|}\int_{B_p^k}\|T_x\|^p dx\right)^{-\frac{1}{p}} \geq \left(\sum_{i=1}^n |\langle x, u_i\rangle|^p dx\right)^{-\frac{1}{p}}.$$

Proof of Theorem 2.3.2. Let $(u_i)_1^n$ and p be as above. For each i let v_i be the unit vector in the direction of u_i. By Lemma 2.3.5,

$$\begin{aligned}
&|\{x \in \mathbb{R}^k : |\langle x_i, u_i\rangle| \leq 1 \text{ for every } i\}|^{\frac{1}{k}} \geq 2\left(\frac{k+p}{k|B_p^k|}\int_{B_p^k}|\langle x, u_i\rangle|^p dx\right)^{-\frac{1}{p}} = \\
&= 2\left(\frac{k+p}{k}\sum_{i=1}^n |u_i|^p . \frac{1}{|B_p^k|}\int_{B_p^k}|\langle x, u_i\rangle|^p dx\right)^{-\frac{1}{p}} \\
&\geq 2\left(\frac{1}{k}\sum_{i=1}^n |u_i|^p . min\left(\{\frac{k+p}{|B_p^k|}\int_{B_p^k}|\langle x, u\rangle|^p dx\right)^{-\frac{1}{p}}\}
\end{aligned}$$

where the minimum is taken over all vectors v of Euclidean length 1. So to complete the proof it suffices to show that for such a vector v,

$$\left(\frac{k+p}{|B_p^k|}\int_{B_p^k}|\langle x, v\rangle|^p dx\right)^{\frac{1}{p}} \leq \begin{cases}\sqrt{\frac{p}{2}} & \text{if} \quad p \geq 2 \\ 2 & \text{if } 1 \leq p < 2.\end{cases}$$

Let $\left(x^{(j)}\right)_1^k$ and $\left(v^{(j)}\right)_1^k$ be the coordinates of the vectors x and v in $\mathbf{R}^k$. For $p \geq 2$, observe that the functions $\left(x^{(j)}v^{(j)}\right)$ on B_p^k form a conditionally symmetric sequence, so by Khintchine's inequality and Hölder's inequality (for $\sum_1^n v^{(j)2} = 1$),

$$\begin{aligned}
&\left(\frac{k+p}{|B_p^k|}\int_{B_p^k}\left|\sum_1^k x^{(j)}v^{(j)}\right|^p dx\right)^{\frac{1}{p}} \\
&\leq \sqrt{\frac{p}{2}}\left(\frac{k+p}{|B_p^k|}\int_{B_p^k}\left(\sum_1^k x^{(j)2}v^{(j)2}\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \\
&\leq \sqrt{\frac{p}{2}}\left(\frac{k+p}{|B_p^k|}\int_{B_p^k}\sum_1^n \left|x^{(j)}\right|^p \cdot v^{(j)2} dx\right)^{\frac{1}{p}} \\
&= \sqrt{\frac{p}{2}}\left(\frac{k+p}{|B_p^k|}\int_{B_p^k}\left|x^{(1)}\right|^p dx\right)^{\frac{1}{p}} \\
&= \sqrt{\frac{p}{2}}.
\end{aligned}$$

For $1 \le p < 2$ it is easily checked that

$$\left(\frac{k+p}{|B_p^k|}\int_{B_p^k} |\langle x,u\rangle|^p dx\right)^{\frac{1}{p}}$$

$$(k+p)^{\frac{1}{p}}\sqrt{\left(\frac{1}{|B_p^k|}\int_{B_p^k} \langle x,u\rangle^2 dx\right)}$$

$$(k+p)^{\frac{1}{p}}\sqrt{\left(\frac{1}{|B_p^k|}\int_{B_p^k} \left(x^{(}1)\right)^2 dx\right)}$$

and the last expression can be (rather roughly) estimated by 2 using standard inequalities involving logarithmically concave functions. □

□

Remark 2. *The proof of Theorem 2.3.2 can be simplified even further if the integration over* B_p^k *is replaced by integration over* S^{k-1} *(and Hölder's inequality applied here). The proof was presented as above because Lemma 5 has some intrinsic interest: for example it may be used to recover Gluskin's precise estimate as follows. Suppose* $m \in \mathbb{N}$ *and the vectors* $(z_i)_1^m \in \mathbb{R}^k$ *satisfy*

$$|z_i| \le \left(\log\left(1+\frac{m}{k}\right)\right)^{-\frac{1}{2}}, \quad 1 \le i \le m.$$

For $\varepsilon > 0$, let $W(\varepsilon)$ be the set

$$\left\{x \in \mathbb{R}^k\colon\ \max_j |x^{(j)}| \le 1, \max_i |\langle x, z_i\rangle| \le \frac{1}{\varepsilon}\right\}:$$

that is, $W(\varepsilon)$ is the intersection of the cube B_∞^k with m "bands" of width at most $\frac{2}{\varepsilon}\sqrt{\log(1+\frac{m}{k})}$. Then $|W(\varepsilon)|^{\frac{1}{k}} \to 2$ as $\varepsilon \to 0$, uniformly in k and m. To see this, apply Lemma 5 with $n = k+m$, the first k, u_i's being the standard basis vectors of $\mathbb{R}^k$ and the remaining m being the vectors $(\varepsilon z_i)_1^m$. If e_j is a standard basis vector,

$$\frac{k+p}{|B_p^k|}\int_{B_p^k} |\langle x, e_j\rangle|^p dx = 1$$

and so Lemma 2.3.5 (and the proof of Theorem 2.3.2) show that for each $p \ge 2, |W(\varepsilon)|^{\frac{1}{k}} \ge 2(1+\frac{m}{k}(\frac{p}{2})^{\frac{p}{2}}\varepsilon^p(\log(1+\frac{m}{k}))^{-\frac{p}{2}})^{-\frac{1}{p}}$ and the latter is at least $\frac{2}{1+\sqrt{e}\cdot\varepsilon}$ if $p = \max(2, 2\log(1+\frac{m}{k}))$. □

As was briefly mentioned earlier, more general estimates than that of Theorem 2 are obtained in [B-P] (for entropy numbers instead of volumes). It is worth noting however that even the argument of Theorem 2 can be used to give the following: there is a constant c so that if $u_1, \ldots, u_n \in \mathbb{R}^k, k \le n$ and $T\colon\ \ell_2^k \to \ell_\infty^n$ is given by $(Tx)_i = \langle u_i, x\rangle,\ 1 \le i \le n$, then the k^{th} entropy number of T satisfies

$$e_k(T) \leq \frac{c\sqrt{p}}{\sqrt{k}} \left(\frac{1}{k} \sum_1^n |u_i|^p \right)^{\frac{1}{p}} \left(1 + \log \frac{n}{k} \right)^{\frac{1}{p}}$$

and hence

$$e_k(T) \leq \frac{ec}{\sqrt{k}} \sqrt{1 + \log \frac{n}{k}} \cdot \|T\|$$

(taking $p = 2(1+\log \frac{n}{k})$). To obtain this one uses Schütt's estimates, [5], for the entropy numbers of the formal identity from ℓ_p^n to ℓ_∞^n in place of the result of Meyer and Pajor, and the dual Sudakov inequality of Pajor and Tomczak, [21] in place of the application of Hölder's inequality.

2.3.3 An application to linear forms.

As stated in the introduction, Vaaler's original result has applications to the geometry of numbers. One such, a sharpened form of Siegel's lemma, is given in [22]. Using the arguments of Bombieri and Vaaler and Theorem 2.3.2, one can obtain the generalisation of their result, contained in Theorem 6, below. Some notation is needed. If A is a $k \times n$ matrix of reals with independent rows ($1 \leq k \leq n$), denote by $v_j = v_j(A)$, $1 \leq j \leq k$, the rows of A. Let $(e_i)_1^n$ be the standard basis of $\mathbb{R}^n$ and denote by c_i, the distance (in the Euclidean norm) of e_i from the span of the v_j's in $\mathbb{R}^n$. (So if A_i is the matrix with $k+1$ rows, $v_1, \ldots, v_n, e_i$ then

$$c_i^2 = \frac{\det(A_i A_i^*)}{\det(AA^*)} \quad \text{for} \quad 1 \leq i \leq n.)$$

Theorem 2.3.6. *Let A be a $k \times n$ matrix with rank k and integral entries. With the notation above, the system $Ax = 0$ admits $n-k$ linearly independent solutions*

$$z^{(r)} = (z_1^{(r)}, \ldots, z_n^{(r)}) \in \mathbb{Z}^n, \ 1 \leq r \leq n-k$$

so that for every $p \geq 2$,

$$\prod_{1 \leq r \leq n-k} \max_i |z_i^{(r)}| \leq D^{-1} \sqrt{\frac{p}{2}} \left(\frac{1}{n-k} \sum_1^n c_i^p \right)^{\frac{1}{p}} \sqrt{\det AA^*}$$

where D denotes the G-C-D of all $k \times k$ determinants extracted from A. ☐

Remark 3. *The principal importance of such a generalisation of Bombieri and Vaaler's result is that it takes into account, more strongly, the form of the matrix A. If the c_i's are all about the same size, then for $p > 2$, the expression*

$$\left(\frac{1}{n-k} \sum c_i^p \right)^{\frac{1}{p}}$$

is small compared with the corresponding expression in which p is replaced by 2.

2.4 Shadows of Convex bodies

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Abstract:
It is proved that if C is a convex body in $\mathbb{R}^n$ then C has an affine image $\widetilde{C}$ (of non-zero volume) so that if P is any 1-codimensional orthogonal projection,

$$|P\widetilde{C}| \geq |\widetilde{C}|^{\frac{n-1}{n}}.$$

It is also shown that there is a pathological body, K, all of whose orthogonal projections have volume about $\sqrt{n}$ times as large as $|K|^{\frac{n-1}{n}}$.

A.M.S. (1980) Subject Classification: 52A20, 52A40

(1)Supported in part by N.S.F. DMS-8807243

2.4.1 Introduction.

The problems discussed in this paper concern the areas of shadows (orthogonal projections) of convex bodies and, to a lesser extent, the surface areas of such bodies. If C is a convex body in $\mathbb{R}^n$ and θ a unit vector, $P_\theta C$ will denote the orthogonal projection of C onto the 1-codimensional space perpendicular to θ. Volumes and areas of convex bodies and their surfaces will be denoted with $|\cdot|$.

The relationship between shadows and surface areas of convex bodies is expressed in Cauchy's well-known formula. For each $n \in \mathbb{N}$, let v_n be the volume of the n-dimensional Euclidean unit ball and let $\sigma = \sigma_{n-1}$ be the rotationally invariant probability on the unit sphere S^{n-1}. Cauchy's formula states that if C is a convex body in $\mathbb{R}^n$ then its surface area is

$$|\partial C| = \frac{nv_n}{v_{n-1}} \int_{S^{n-1}} |P_\theta C| d\sigma(\theta).$$

The classical isoperimetric inequality in $\mathbb{R}^n$ states that any body has surface area at least as large as an Euclidean ball of the same volume. The first section of this paper is devoted to the proof of a "local" isoperimetric inequality showing that all bodies have large shadows rather than merely large surface area (or average shadow). The principal motivation for this result is its relationship to a conjecture of Vaaler and the important problems surrounding it. This theorem and its connection with Vaaler's conjecture are described at the beginning of Section 1.

An important role is played in the theory of convex bodies by the so-called "projection body" of a convex body. It is easily seen, by considering polytopes, that for every convex body $C \subset \mathbb{R}^n$, there is a Borel measure μ on S^{n-1} so that for each $\theta \in S^{n-1}$,

$$|P_\theta C| = \int_{S^{n-1}} |\langle \theta, \phi \rangle| d\mu(\phi).$$

Hence, there is a norm $\|\cdot\|$ on $\mathbb{R}^n$, with $\|\theta\| = |P_\theta C|$ for each $\theta \in S^{n-1}$, with respect to which $\mathbb{R}^n$ is isometrically isomorphic to a subspace of L_1. The unit ball of this norm is a symmetric convex body which will be denoted $\Pi^*(C)$. The map Π^*, from the collection of convex bodies in $\mathbb{R}^n$ to the collection of unit balls of representations of n-dimensional subspaces of L_1 on $\mathbb{R}^n$, has been extensively studied: see e.g.[23]. The restriction of Π^* to the class of centrally symmetric convex bodies was shown to be injective by Aleksandrov: if C and D are centrally symmetric convex bodies and $|P_\theta C| = |P_\theta D|$ for all $\theta \in S^{n-1}$ then $C = D$. If the condition of central symmetry is dropped, C and D may not even be congruent: the Rouleaux triangle in $\mathbb{R}^2$ has all 1-dimensional shadows equal in length to those of some disc.

The map Π^* was shown to be surjective by Minkowski. What Minkowski's proof actually gives (at least in the context of polytopes) is the result stated in Section 2 as Lemma 6. (This rather detailed statement of Minkowski's theorem will be needed for the construction of a pathological body with large shadows.)

An important observation of Petty [24], is that if T is a linear operator on $\mathbb{R}^n$ of determinant 1 then, for every C,

$$\Pi^*(TC) = T(\Pi^*(C)). \tag{1}$$

Motivated in part by Aleksandrov's theorem on the injectivity of Π^*, Shephard asked whether, if C and D are centrally symmetric convex bodies with

$$|P_\theta C| \geq |P_\theta D| \quad \text{for all} \quad \theta \in S^{n-1}$$

then necessarily $|C| \geq |D|$. This question was answered in the negative by Petty and Schneider independently in [24] and [25]. (The corresponding question for sections rather than shadows was posed in [26] and answered (again in the negative) by [27].) The second section of this paper contains a strongly negative answer to Shephard's question. It will be shown that a "random" n-dimensional subspace of ℓ_∞^{2n} has a unit ball, all of whose shadows are very large compared with those of a Euclidean ball of the same volume. Such examples suggest that the Shephard problem is less delicate than the Busemann-Petty problem for sections: it is an important open question as to whether there are highly pathological examples for the latter problem. This question is usually referred to as the slicing problem.

2.4.2 A local isoperimetric inequality.

In [28], Vaaler conjectured that for every $n \in \mathbb{N}$, every symmetric convex body $C \subset \mathbb{R}^n$ and every $k < n$, there is an affine image TC of C (for some automorphism T of $\mathbb{R}^n$) so that for every k-dimensional subspace H of $\mathbb{R}^n$

$$|H \cap TC| \geq |TC|^{\frac{k}{n}}.$$

(Vaaler actually conjectured something slightly stronger, which is false for small values of k.) This conjecture strengthens the slicing mentioned above, namely: there exists a $\delta > 0$ so that for each n and C there is a 1-codimensional subspace H of $\mathbb{R}^n$ with

$$|H \cap C| \geq \delta |C|^{\frac{n-1}{n}}.$$

The case $k = 1$ of Vaaler's conjecture (for arbitrary n and C) was proved by the present author in [B$_1$], where the result is stated as a volume ratio estimate. The proof for $k = 1$ really estimates volumes of 1-dimensional shadows and then uses the fact that the smallest 1-dimensional section of a convex body is its smallest 1-dimensional shadow. Since minimal sections and minimal shadows are not identical for subspaces of dimension larger than 1, such an argument cannot be employed if $k > 1$: but it is natural to ask whether Vaaler's conjecture can be proved for shadows (of dimension other than 1) independently of the outstanding problem for sections. The principal result of this paper deals with the most important case,
$k = n - 1$.

Theorem 2.4.1. *Let C be a convex body in $\mathbb{R}^n$. There is an affine image $\widetilde{C}$ of C (with non-zero volume) so that for each unit vector $\theta \in \mathbb{R}^n$,*

$$|P_\theta \widetilde{C}| \geq |\widetilde{C}|^{\frac{n-1}{n}}.$$

The result is exactly best possible as shown by the cube.

The proof of Theorem 1 uses the well-known theorem of John [29] which characterises ellipsoids of minimal volume containing convex bodies. This result is stated here as a lemma.

Lemma 2.4.2. *Let K be a symmetric convex body in $\mathbb{R}^n$. The ellipsoid of minimal volume containing K is the Euclidean unit ball B_2^n if and only if K is contained in B_2^n and there are Euclidean unit vectors $(u_i)_1^m$ (for some $m \in \mathbb{N}$) on the boundary ∂K of K and positive numbers $(c_i)_1^m$ so that*

$$\sum_1^m c_i u_i \otimes u_i = I_n.$$

(Here, $u_i \otimes u_i$ is the usual rank-1 orthogonal projection onto the span of u_i and I_n is the identity on $\mathbb{R}^n$.) The identity above states that the u_i's are distributed rather like an orthonormal basis in that for each $x \in \mathbb{R}^n$,

$$|x|^2 = \sum_1^m c_i \langle u_i, x\rangle^2.$$

The equality of the traces of the operators appearing above shows that $\sum\limits_1^m c_i = n$.

Theorem 2.4.1 will be deduced from the following, which is little more than an affine invariant reformulation.

Theorem 2.4.3. *Suppose C is a convex body in $\mathbb{R}^n$, $(u_i)_1^m$ a sequence of unit vectors in $\mathbb{R}^n$ and $(c_i)_1^m$ a sequence of positive numbers for which*

$$\sum_1^m c_i u_i \otimes u_i = I_n.$$

For each i, let P_i be the orthogonal projection P_{u_i} along u_i. Then

$$|C|^{n-1} \leq \prod_1^m |P_i C|^{c_i}.$$

There is an obvious relationship between Theorem 2.4.3,2.4.2, 2.4.1 . is closely related to an inequality of Brascamp and Lieb [30] which has been used in several places by this author, [31] and [32]. Theorem 2.4.3 and generalisations of it were conjectured in [30] (in a different form). The special case of Theorem 2.4.3 in which the u_i's form an orthonormal basis (in which case, necessarily, $c_i = 1$ for $1 \leq i \leq m = n$) was proved by Loomis and Whitney [33]. Theorem 3 can be regarded as an isoperimetric inequality in that it estimates the volume of a body in terms of an average of volumes of its shadows: in this case, a geometric average. The key point is that the "number" of shadows involved is small enough that the local isoperimetric inequality of Theorem 2.4.1 can be deduced.

Proof of Theorem 2.4.1. Because of the intertwining property of Π^* with linear transformations (1), there is an affine image $\widetilde{C}$ of C so that the ellipsoid of minimal volume containing $\Pi^*(\widetilde{C})$ is the Euclidean ball B_2^n. This ensures that

$$|P_\theta(\widetilde{C})| \geq 1$$

for every unit vector $\theta \in \mathbb{R}^n$ and, by Lemma 2.4.2, that there are unit vectors $(u_i)_1^m$ and positive numbers $(c_i)_1^m$ so that

$$|P_{u_i}\widetilde{C}| = 1 \quad \text{for each} \quad i$$

and $\sum_1^m c_i u_i \otimes u_i = I_n$.

Now, from Theorem 2.4.3,

$$|\widetilde{C}|^{n-1} \leq \prod_1^m |P_{u_i}\widetilde{C}|^{c_i} = 1.$$

□ □

The proof of Theorem 2.4.3 uses Minkowski's inequality for mixed volumes to establish a duality between the 1-codimensional problem to be solved and a 1-dimensional problem. The relevant information on mixed volumes is included here for completeness.

For a fixed n, let $\mathcal{C} = \mathcal{C}_n$ be the set of compact, convex subsets of $\mathbb{R}^n$. $\mathcal{C}$ can be regarded as a convex cone under Minkowski addition and multiplication by non-negative scalars. A crucial theorem of Minkowski states that there is a symmetric n-positive-linear form

$$V\colon \underbrace{\mathcal{C} \times \cdots \times \mathcal{C}}_{n \text{ times}} \longrightarrow [0, \infty)$$

whose diagonal is volume: i.e. V is positive linear in each of its n arguments and, for each $C \in \mathcal{C}$,

$$|C| = V(C, \dots, C).$$

The values of V are called mixed volumes. As a consequence of Minkowski's theorem, the volume $|C + tD|, (t \in [0, \infty))$ can be expanded as a polynomial in t,

$$|C + tD| = \sum_0^n \binom{n}{k} v_{n-k}(C, D) t^k \text{ where } v_{n-k}(C, D) = V\,\underbrace{(C, \dots, C)}_{n-k}\,\underbrace{(D, \dots, D)}_{k}$$

and is called the $n - k^{\text{th}}$ mixed volume of C and D.

The Brunn-Minkowski inequality states that

$$|C + tD|^{\frac{1}{n}}$$

is a concave function of t (on $[0, \infty)$). Differentiation of (2) at $t = 0$ gives Minkowski's inequality

$$|C|^{\frac{n-1}{n}} |D|^{\frac{1}{n}} \leq v_{n-1}(C, D). \tag{3}$$

If $D = B_2^n$ is the Euclidean unit ball, (3) is the classical isoperimetric inequality. Inequality (3) will be used here with an appropriate choice of D.

A Minkowski sum of line segments

$$\sum_1^m [-x_i, x_i] = \left\{ x \in \mathbb{R}^n \colon\ x = \sum_1^m \lambda_i x_i \ \text{ for some sequence } (\lambda_i)_1^m \text{ with } |\lambda_i| \leq 1, 1 \leq i \leq m \right\}$$

is called a zonotope. It is easily checked that if u is a unit vector and $D = [-u, u] = \{x \in \mathbb{R}^n \colon\ x = \lambda u \text{ for some } \lambda \in [-1, 1]\}$ then

$$v_{n-1}(C, D) = \frac{2}{n} |P_u C|$$

for any convex body C. So if Z is the zonotope

$$Z = \sum_1^m \alpha_i [-u_i, u_i]$$

with $(u_i)_1^m$ a sequence of unit vectors and $(\alpha_i)_1^m$ a sequence of positive numbers,

$$v_{n-1}(C, Z) = \frac{2}{n} \sum_1^m \alpha_i |P_{u_i} C|. \tag{4}$$

There is equality in Minkowski's inequality (3) if $C = D$ and so with Z as before

$$|Z| = \frac{2}{n}\sum_{1}^{m} \alpha_i |Pu_i Z|. \tag{5}$$

This identity is usually called the volume formula for zonotopes. A simple induction argument can be used to obtain an expression for $|Z|$ in terms of the α_i's and determinants of square matrices formed from the u_i's. In the following lemma, a similar inductive argument is used to obtain an estimate for $|Z|$ which is easier to use than the actual value.

Lemma 2.4.4. *Let $(u_i)_1^m$ be a sequence of unit vectors in $\mathbb{R}^n, (c_i)_1^m$ a sequence of positive numbers with*

$$\sum_{1}^{m} c_i u_i \otimes u_i = I_n$$

and $(\alpha_i)_1^m$ another sequence of positive numbers. If

$$z = \sum_{1}^{m} \alpha_i [-u_i, u_i] \text{ then } |z| \geq 2^n \prod_{1}^{m} \left(\frac{\alpha_i}{c_i}\right)^{c}_{i}$$

Proof. The proof uses induction on the dimension n. For $n = 1$,

$$\begin{aligned} |Z| &= 2\sum_{1}^{m} \alpha_i \\ &= 2\sum_{1}^{m} c_i \Big(\frac{\alpha_i}{c_i}\Big) \\ &\geq 2\prod_{1}^{m} \Big(\frac{\alpha_i}{c_i}\Big)^{c_i}. \end{aligned}$$

by the AM-GM inequality since $\sum_{1}^{m} c_i = 1$, when $n = 1$.

For larger n, the volume formula (5) shows that with $P_i = P_{u_i}$

$$\begin{aligned} |Z| &= \frac{2}{n}\sum_{1}^{m} \alpha_i |P_i Z| \\ &= 2\sum_{1}^{m} \frac{c_i}{n}\frac{\alpha_i}{c_i}|P_i Z| \\ &\geq 2\prod_{1}^{m} \Big(\frac{\alpha_i}{c_i}|P_i Z|\Big)^{\frac{c_i}{n}} \end{aligned}$$

since $\sum_{1}^{m} c_i = n$.

Now, for each fixed i, P_iZ is a zonotope with summands

$$\alpha_j[-P_i(u_j), P_i(u_j)], \qquad 1 \le j \le m$$

and contained in the $(n-1)$-dimensional space $P_i(\mathbb{R}^n)$. For each i and j let

$$\gamma_{ij} = |P_i(u_j)| = |P_j(u_i)|$$

(so $\gamma_{ij}^2 = 1 - \langle u_i, u_j\rangle^2$). Then for each i,

$$P_iZ = \sum_{j=1}^{m} \alpha_j \gamma_{ij}[-v_{ij}, v_{ij}]$$

where v_{ij} is the unit vector in the direction of P_iu_j (or any direction if $\gamma_{ij} = 0$). Now, for each i,

$$\begin{aligned} P_i &= \sum_{j=1}^{m} c_j P_iu_j \otimes P_iu_j \\ &= \sum_{j=1}^{m} c_j \gamma_{ij}^2 v_{ij} \otimes v_{ij} \end{aligned} \tag{2.1}$$

and P_i acts as the identity on $P_i(\mathbb{R}^n)$. So, by the inductive hypothesis,

$$|P_iZ| \ge 2^{n-1} \prod_{j=1}^{m} \left(\frac{\alpha_j\gamma_{ij}}{c_j\gamma_{ij}^2}\right)^{c_j\gamma_{ij}^2}$$

where it is understood that if $\gamma_{ij} = 0$, the j^{th} factor is 1: (so in particular, the i^{th} factor is 1).

Substitution of the inequalities for each i into 2.1 shows that

$$|Z| \ge 2^n \left(\prod_{i,j=1}^{m} \left(\frac{\alpha_i}{c_i}\right)^{c_i} \left(\frac{\alpha_j}{c_j\gamma_{ij}}\right)^{c_ic_j\gamma_{ij}^2} \right)^{\frac{1}{n}}$$

and this expression is at least

$$2^n \prod_{i=1}^{m} \left(\frac{\alpha_i}{c_i}\right)^{c_i}$$

because $\frac{1}{\gamma_{ij}} \ge 1$ for all i and j and, for each j,

$$\sum_{i=1}^{m} c_i\gamma_{ij}^2 = \sum_{i=1}^{m} c_i(1 - \langle u_i, u_i\rangle^2) = n - 1.$$

$\square$

Proof of Theorem 2.4.3. Let $C, (u_i)_1^m, (c_i)_1^m$ and $(P_i)_1^m$ be as in the theorem's statement. For each i, set

$$\alpha_i = \frac{c_i}{|P_i C|}.$$

With $Z = \sum\limits_1^m \alpha_i[-u_i, u_i]$, (3) and (4) above, show that

$$\begin{aligned} |C|^{n-1} &\le |Z|^{-1}\left(\frac{2}{n}\sum_1^m \alpha_i |P_i C|\right)^n \\ &= |Z|^{-1}\left(\frac{2}{n}\sum_1^m c_i\right)^n \\ &= 2^n |Z|^{-1} \end{aligned}$$

since $\sum\limits_1^m c_i = n$.

Now by Lemma 2.4.4,

$$|C|^{n-1} \le 2^n \left(2^n \prod_{i=1}^m \left(\frac{\alpha_i}{c_i}\right)^{c_i}\right)^{-1} = \prod_{i=1}^m |P_i C|^{c_i}.$$

□

Remark 4. *The lower estimate for volumes of zonotopes given by Lemma 4 could have been obtained from the volume ratio estimates proved in the author's paper [32] together with Reisner's reverse Santalo inequality for zonoids, [34].*

Remark 5. *For every convex body C in $\mathbb{R}^n$, there is an affine image $\widetilde{C}$ of C with*

$$|P_\theta \widetilde{C}| \le M\sqrt{n}|\widetilde{C}|^{\frac{n-1}{n}}$$

for each unit vector θ (M being an absolute constant). This estimate depends upon the fact that subspaces of L_1 have uniformly bounded volume ratios. It seems likely that M could be taken to be 1 for symmetric convex bodies C (the cube again being extremal). The estimates in [31] show that M can be taken to be $\frac{2\sqrt{e}}{\pi} \approx 1.05$ in this case.

2.4.3 A remark on the Shephard problem.

This section contains a strongly negative solution to the Shephard problem described in the introduction. Petty and Schneider ([24] and [25]) constructed pairs of bodies C and D in $\mathbb{R}^n$ so that

$$|P_\theta D| \le |P_\theta C| \quad \text{for all} \quad \theta \in S^{n-1}, \tag{7}$$

but $|D| > |C|$. Schneider also showed that the conclusion $|D| \leq |C|$ does hold if C is a zonoid (a limit, in the Hausdorff metric, of zonotopes). A little more generally, if C and D are convex bodies satisfying (7) and $Z = \sum_1^m \alpha_i[-u_i, u_i]$ is a zonotope included in C (with $u_i \in S^{n-1}, 1 \leq i \leq m$), then, rather as in the proof of Theorem 3,

$$\begin{aligned}|D|^{\frac{n-1}{n}}|Z|^{\frac{1}{n}} &\leq v_{n-1}(D, Z) = \sum_1^m \alpha_i |P_{u_i} D| \\ &\leq \sum_1^m \alpha_i |P_{u_i} C| = v_{n-1}(C, Z) \\ &\leq v_{n-1}(C, C) = |C|\end{aligned}$$

where the last inequality is a consequence of the monotonicity of mixed volumes: (this particular case is obvious from the fact that if $Z \subset C$,

$$C + tZ \subset (1 + t)C$$

for all $t \geq 0$).

Hence,

$$|D| \leq \left(\frac{|C|}{|Z|}\right)^{\frac{1}{n-1}} |C|.$$

Since every convex body C contains a zonoid Z with $\frac{|C|}{|Z|}$ not too large, the last inequality shows that under hypothesis (7), one does have

$$|D| \leq \frac{3}{2}\sqrt{n}|C|.$$

Apart from a constant factor, it turns out that this is the most that can be said. The Euclidean ball of volume 1 in $\mathbb{R}^n$ has shadows of 1-codimensional volume about $\sqrt{e}$ (as $n \to \infty$): in Theorem 5 it is shown that there is a body of volume 1 in $\mathbb{R}^n$, all of whose shadows have volume about $\sqrt{n}$.

Theorem 2.4.5. *There is a constant $\delta > 0$ so that for each $n \in \mathbb{N}$, there is a symmetric convex body K in $\mathbb{R}^n$ satisfying,*

$$|P_\theta K| \geq \delta\sqrt{n}|K|^{\frac{n-1}{n}}$$

for every unit vector $\theta \in \mathbb{R}^n$.

The proof of Theorem 2.4.5 depends heavily upon the theorem of Minkowski on the existence of bodies with given projections. An appropriately detailed statement of this theorem is given as Lemma 2.4.6 below: the following notation is needed. For a sequence $(u_i)_1^m$ of unit vectors spanning $\mathbb{R}^n$ and a sequence $(\gamma_i)_1^m$ of positive numbers let $\mathcal{F} = \mathcal{F}((u_i), (\gamma_i))$ be the family of convex bodies of the form

$$\{x \in \mathbb{R}^n\colon\ |\langle x, u_i\rangle| \le t_i, \quad 1 \le i \le m\}$$

indexed by sequences $(t_i)_1^m$ of positive reals satisfying

$$\sum_1^m \gamma_i t_i = 1.$$

Lemma 2.4.6. *With the above notation, $\mathcal{F}$ has an unique element of maximal volume, K (say), satisfying*

$$|P_\theta K| = \frac{n|K|}{2} \sum_1^m \gamma_i |\langle u_i, \theta\rangle|$$

for each $\theta \in S^{n-1}$. ☐

The body that satisfies the conclusion of Theorem 2.4.5 will be the unit ball of a "random" n-dimensional subspace of ℓ_∞^{2n}. Such subspaces are known to have many pathological properties stemming from the fact that the ℓ_1^{2n} and ℓ_2^{2n} norms are well-equivalent on such spaces: this was proved in [35]. For the history of this result and its many extensions, see e.g. [36]. The form of the result needed here is given as a Lemma.

Lemma 2.4.7. *There is a $\delta > 0$ so that for $n \in \mathbb{N}$ there are unit vectors $u_1, \ldots, u_{2n}$ in $\mathbb{R}^n$ with*

$$\sum_1^{2n} |\langle x, u_i\rangle| \ge \delta\sqrt{n}|x|$$

for every vector $x \in \mathbb{R}^n$. ☐

The last lemma that will be needed is a result of Vaaler [28], concerning volumes of sections of the cube in $\mathbb{R}^n$. The form required here is the following.

Lemma 2.4.8. *Let $(u_i)_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$. Then the volume of the symmetric convex body with these vectors as boundary functionals satisfies.*

$$|\{x \in \mathbb{R}^n\colon\ |\langle x, u_i\rangle| \le 1, \quad 1 \le i \le m\}|^{\frac{1}{n}} \ge 2\sqrt{\frac{n}{m}}.$$ ☐

5.

Proof of Theorem 2.4.5. Let $(u_i)_1^{2n}$ be a sequence of vectors with the property described in Lemma 2.4.7. Take $m = 2n$ and apply Lemma 2.4.6 with $\gamma_i = \frac{1}{m}, 1 \le i \le m$.

The body K of maximal volume in the family $\mathcal{F}$ satisfies

$$|P_\theta K| = \frac{n|K|}{2m} \sum_1^m |\langle u_i, \theta\rangle| \ge \frac{|K|}{4} \cdot \delta\sqrt{n}$$

for every unit vector θ. Now, $\mathcal{F}$ also contains the body

$$C = \{x \in \mathbb{R}^n \colon \ |\langle x, u_i \rangle| \leq 1, \quad 1 \leq i \leq m\}.$$

So (by the maximality of K), $|K| \geq |C|$. But, by Lemma 2.4.8,

$$|C|^{\frac{1}{n}} \geq 2\sqrt{\frac{n}{m}} = \sqrt{2}$$

and hence $|K|^{\frac{1}{n}} \geq \sqrt{2}$. Therefore, for every unit vector θ,

$$|P_\theta K| \geq \frac{\delta\sqrt{n}}{2\sqrt{2}}|K|^{\frac{n-1}{n}}. \qquad \square$$

$\square$

Remark 6. *The above argument can be extended slightly to estimate surface area to volume ratios of subspaces of* ℓ_∞*. If* $iq(X)$ *is the isoperimetric quotient of the finite-dimensional normed space* X *(see [37] for definitions) normalised so that* $iq(\ell_2^n) = 1$ *then for every* n*-dimensional subspace* X *of* ℓ_∞^m*,*

$$iq(X) \geq \frac{\delta\sqrt{n}}{\sqrt{1 + \log\frac{m}{n}}}$$

for some absolute constant $\delta > 0$.

2.5 Volume ratios and a reverse isoperimetric inequality

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Abstract:
It is shown that if C is an n-dimensional convex body then there is an affine image $\widetilde{C}$ of C for which

$$\frac{|\partial\widetilde{C}|}{|\widetilde{C}|^{\frac{n-1}{n}}}$$

is no larger than the corresponding expression for a regular n-dimensional "tetrahedron". It is also shown that among n-dimensional subspaces of L_p (for each $p \in [1,\infty]$), ℓ_p^n has maximal volume ratio.

A.M.S. (1980) Subject Classification: 52A20

(1)Supported in part by N.S.F. DMS-8807243

2.5.1 Introduction.

The famous isoperimetric inequality in $\mathbb{R}^n$ states that among bodies of a given volume, the Euclidean balls have least surface area. Measurable sets of finite volume may have infinite "surface area" and, if $n \geq 2$, even convex bodies of a given volume may have arbitrarily large surface area if they are very flat. Nevertheless, classical inequalities such as the isoperimetric inequality do admit of reverse forms: the important reverse Santalo and reverse Brunn-Minkowski inequalities of [38] and [39] are examples.

Probably the most natural way to reverse the isoperimetric inequality is to consider classes of affinely equivalent convex bodies rather than individual bodies. The inequality between volume and surface area is proved only for one representative of each class; the "least flat" member of that class.

Modulo affine transformations it will be shown that among all convex bodies in $\mathbb{R}^n$, the n-dimensional tetrahedron has "largest" surface area for a given volume, while among symmetric convex bodies, the cube is extremal. More precisely, the principal theorems proved in this paper are the following: (volume and area are denoted throughout by $|\cdot|$).

Theorem 2.5.1. *Let C be a convex body in $\mathbb{R}^n$ and T a regular n-dimensional tetrahedron (solid simplex). Then there is an affine image $\widetilde{C}$ of C satisfying*

$$|\widetilde{C}| = |T| \quad \text{and}$$
$$|\partial\widetilde{C}| \leq |\partial T|.$$

Theorem 2.5.2. *If C is a symmetric convex body in $\mathbb{R}^n$ and Q an n-dimensional cube then there is an affine image $\widetilde{C}$ of C satisfying*

$$|\widetilde{C}| = |Q| \quad \text{and}$$
$$|\partial\widetilde{C}| \leq |\partial Q|.$$

Theorems 2.5.1 and 2.5.2 are proved in Section 1 below. The upper bounds for surface area that are needed depend upon volume ratio estimates. The volume ratio of an n-dimensional convex body C is

$$vr(C) = \Big(\frac{|C|}{|\mathcal{E}|}\Big)^{\frac{1}{n}}$$

where $\mathcal{E}$ is the ellipsoid of maximal volume included in C. Similarly, if X is an n-dimensional normed space, $vr(X)$ is defined to be $vr(C)$ for any convex body C which is the unit ball of X in some representation of X on $\mathbb{R}^n$. Section 2 of this paper deals with a further question about volume ratios. It was proved in [38] that the volume ratio of a finite-dimensional normed spaced can be bounded above, solely in terms of the cotype-2 constant of the space: (see e.g. [40] for definitions). In particular, finite-dimensional subspaces of L_1 have uniformly bounded volume ratios. An isometric form of this result is proved below: it is shown that for each $p \in [1, \infty]$, ℓ_p^n has maximal volume ratio among n-dimensional subspaces of L_p.

This paper constitutes a sequel to the paper [41] which appeared recently.

2.5.2 The reverse isoperimetric inequality.

There are (at least) two ways in which to couch reverse forms of the classical inequalities for convex bodies. In the case of the reverse Santalo inequality, the expression to be estimated is $|C||C^0|$. This expression, the volume product for a symmetric convex body and its polar is invariant under linear transformation of C. So there are bodies for which the expression is minimal. Surface area does not behave well under linear transformations. Although there are affine invariants which measure surface area it seems natural to reverse the isoperimetric inequality by choosing representatives of affine equivalence classes of bodies, as described in the introduction: (for the affine invariant problem, see the appendix at the end of this paper).

For many of the classical inequalities involving convex bodies, the extremal bodies are ellipsoids or Euclidean balls and often this means that the inequalities can be proved by well-known symmetrisation techniques. For the reverse inequalities, one expects extremal bodies such as cubes or tetrahedra: because of this, classical symmetrisation methods do not seem to be readily applicable.

As was mentioned earlier, Theorems 2.5.1 and 2.5.2 are proved via volume ratio estimates. Two well-known theorems of John [42], characterise ellipsoids of maximal volume contained in convex bodies. These are stated here as lemmas.

Lemma 2.5.3. *Let C be a symmetric convex body in $\mathbb{R}^n$. The ellipsoid of maximal volume in C is the Euclidean unit ball B_2^n, if and only if C contains B_2^n and there is a sequence $(u_i)_1^m$ of contact points between B_2^n and ∂C (i.e. unit vectors on the boundary of C) and a sequence $(c_i)_1^m$ of positive numbers so that*

$$\sum_{1}^{m} c_i u_i \otimes u_i = I_n. \tag{1}$$

Here, $u_i \otimes u_i$ is the rank-1 orthogonal projection onto the span of u_i and I_n is the identity operator on $\mathbb{R}^n$. Condition (1) shows that the u_i's behave like an orthonormal basis to the extent that for each $x \in \mathbb{R}^n$,

$$|x|^2 = \sum_{1}^{m} c_i \langle u_i, x\rangle^2.$$

The equality of the traces in (1) shows that

$$\sum_{1}^{m} c_i = n.$$

Lemma 2.5.4. *Let C be a convex body in $\mathbb{R}^n$ (not necessarily symmetric). The ellipsoid of maximal volume in C is B_2^n, if and only if C contains B_2^n and there are contact points $(u_i)_1^m$ and positive numbers $(c_i)_1^m$ so that*

a) $\sum_{1}^{m} c_i u_i \otimes u_i = I_n$ *and*

b) $\sum_{1}^{m} c_i u_i = 0$

Theorems 2.5.1 and 2.5.2 are proved by combining the theorems of John with a generalised convolution inequality of Brascamp and Lieb, [43]. A “normalised” form of this inequality was introduced in the author’s previous paper [41]. The normalisation is motivated by the theorems of John: its principal advantage is that it “automatically” calculates the best possible constant in the inequality. The theorem of Brascamp and Lieb is stated here as a lemma.

Lemma 2.5.5. *Let $(u_i)_1^m$ be a sequence of unit vectors in $\mathbb{R}^n$ and $(c_i)_1^m$ a sequence of positive numbers so that*

$$\sum_{1}^{m} c_i u_i \otimes u_i = I_n.$$

For each i, let $f_i\colon \mathbb{R} \to [0, \infty)$ be integrable. Then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(\langle u_i, x\rangle)^{c_i} dx \le \prod_{i=1}^{m} \left(\int_{\mathbb{R}} f_i \right)^{c_i}.$$

There is equality in Lemma 2.5.5 if the f_i's are identical Gaussian densities or if the u_i's form an orthonormal basis of $\mathbb{R}^n$ (and in some other cases). Lemma 2.5.5 is a generalisation of Young’s convolution inequality with best possible constant, proved independently by Beckner, [44].

It is relatively simple to combine Lemmas 2.5.3 and 2.5.5 to show that among symmetric convex bodies in $\mathbb{R}^n$, the cube has exactly maximal volume ratio. (This was done in [41] and the result is quoted below as the $p = \infty$ case of Theorem 2.5.7.) Thus, if C is a symmetric convex body whose ellipsoid of maximal volume if B_2^n, then $|C| \leq 2^n$. From this it is easy to deduce Theorem 2.5.2.

Proof of Theorem 2.5.2. Let C be a symmetric convex body in $\mathbb{R}^n$. It is required to show that some affine image $\widetilde{C}$ of C satisfies

$$|\partial \widetilde{C}| \leq 2n|\widetilde{C}|^{\frac{n-1}{n}}$$

since these expressions are equal if $\widetilde{C}$ is a cube in $\mathbb{R}^n$. Choose $\widetilde{C}$ so that its ellipsoid of maximal volume is B_2^n. By the remark above, $|\widetilde{C}| \leq 2^n$. But, since $B_2^n \subset \widetilde{C}$,

$$\begin{aligned} |\partial \widetilde{C}| &= \lim_{\varepsilon \to 0} \frac{|\widetilde{C} + \varepsilon B_2^n| - |\widetilde{C}|}{\varepsilon} \leq \lim_{\varepsilon \to 0} \frac{|\widetilde{C} + \varepsilon \widetilde{C}| - |\widetilde{C}|}{\varepsilon} = \lim_{\varepsilon \to 0} |\widetilde{C}| \cdot \frac{(1+\varepsilon)^n - 1}{\varepsilon} \\ &= n|\widetilde{C}| = n|\widetilde{C}|^{\frac{n-1}{n}} |\widetilde{C}|^{\frac{1}{n}} \leq 2n|\widetilde{C}|^{\frac{n-1}{n}} \end{aligned}$$

□

In exactly the same way, Theorem 1 may be deduced from the volume ratio estimate given by Theorem 1′ below. (In each case the argument loses nothing, because each of the cube and tetrahedron has the property that all its faces touch its ellipsoid if maximal volume.)

Theorem 2.5.6 (Theorem 1′.)**.** *Among all convex bodies in $\mathbb{R}^n$, n-dimensional tetrahedra have maximal volume ratio.*

Proof. The n-dimensional regular tetrahedron that circumscribes B_2^n has volume

$$\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!}.$$

So, it suffices to prove that if C is a convex body whose maximal ellipsoid is B_2^n then

$$|C| \leq \frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!}.$$

By Lemma 4 there are unit vectors $(u_i)_1^m$ on ∂C and positive numbers $(c_i)_1^m$ so that

$$\sum_1^m c_i u_i \otimes u_i = I_n \quad \text{and} \tag{2}$$

$$\sum_1^m c_i u_i = 0. \tag{3}$$

Since the u_i's are contact points of B_2^n and ∂C,

$$C \subset \{x \in \mathbb{R}^n \colon \langle x, u_i \rangle \le 1, 1 \le i \le m\} = K \qquad \text{(say)}.$$

It will be shown that K has volume no more than

$$\frac{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}{n!}.$$

Now, there is equality in Lemma 2.5.3 if the vectors appearing in its statement are orthogonal. The key to the following estimate is the construction of a new sequence of vectors $(v_i)_1^m$ in $\mathbb{R}^{n+1}$ which would be orthogonal in the extreme case in which K is a regular tetrahedron. The estimate follows from an application of Lemma 5 to a family of functions whose product is supported on a cone in $\mathbb{R}^{n+1}$ whose cross-sections are similar to K.

Regard $\mathbb{R}^{n+1}$ as $\mathbb{R}^n \times \mathbb{R}$. For each i, let

$$v_i = \sqrt{\frac{n}{n+1}}\Big(-u_i, \frac{1}{\sqrt{n}}\Big) \in \mathbb{R}^{n+1}$$

and

$$d_i = \frac{n+1}{n} c_i.$$

Then, for each i, v_i is a unit vector and the identities (2) and (3) above, together ensure that

$$\sum_1^m d_i v_i \otimes v_i = I_{n+1}.$$

Also, $\sum\limits_1^m d_i = n+1$. For each i, define $f_i \colon \ \mathbb{R} \to [0, \infty)$ by

$$f_i(t) = \begin{cases} e^{-t} & \text{if} \quad t \ge 0 \\ 0 & \text{if} \quad t < 0. \end{cases}$$

Finally, for $x \in \mathbb{R}^{n+1}$ set

$$F(x) = \prod_{i=1}^m f_i(\langle v_i, x \rangle)^{d_i}.$$

By Lemma 2.5.5,

$$\int_{\mathbb{R}^{n+1}} F(x) dx \le \prod_{i=1}^m \Big(\int_{\mathbb{R}} f_i \Big)^{d_i} = 1. \tag{4}$$

Now, suppose $x = (y, r) \in \mathbb{R}^n \times \mathbb{R}$. For each i,

$$\langle v_i, x \rangle = \frac{r}{\sqrt{n+1}} - \sqrt{\frac{n}{n+1}} \langle u_i, y \rangle.$$

Since $\sum_1^m c_i u_i = 0$, there is some j (depending upon y) for which $\langle u_j, y\rangle \geq 0$. Hence, if $r < 0$, $\langle v_j, x\rangle < 0$ for some j and so $F(x) = 0$. On the other hand, if $r \geq 0$ then $F(x)$ is non-zero precisely if for every i

$$\langle u_i, y\rangle \leq \frac{r}{\sqrt{n}};$$

in this case

$$\begin{aligned}
F(x) &= \exp\Big(-\sum_1^m d_i\Big[\frac{r}{\sqrt{n+1}} - \sqrt{\frac{n}{n+1}}\langle u_i, y\rangle\Big]\Big)\\
&= \exp\Big(-\sqrt{n+1}\cdot r + \sqrt{\frac{n+1}{n}}\Big\langle \sum_1^m c_i u_i, y\Big\rangle\Big)\\
&= \exp(-\sqrt{n+1}\cdot r).
\end{aligned}$$

Thus, for each $r \geq 0$, the integral of F over the hyperplane $\{x\colon\ x_{n+1} = r\}$ is

$$e^{-\sqrt{n+1}r}\Big|\frac{r}{\sqrt{n}}K\Big| = e^{-\sqrt{n+1}r}\Big(\frac{r}{\sqrt{n}}\Big)^n|K|.$$

Therefore, from (4),

$$1 \geq |K|\int_0^\infty e^{-\sqrt{n+1}r}\Big(\frac{r}{\sqrt{n}}\Big)^n dr = \frac{|K|\cdot n!}{n^{\frac{n}{2}}(n+1)^{\frac{n+1}{2}}}$$

as required. □

Remark 7. *It is possible to obtain estimates on outer volume ratio, which involves the ellipsoid of minimal volume containing a body C, by using a reverse form of the Brascamp and Lieb inequality. Among symmetric convex bodies, the n-dimensional "octahedron" is extremal, and among all convex bodies, the tetrahedron.*

2.5.3 Volume ratios of subspaces of L_p.

In [41], the author showed that among n-dimensional normed spaces, ℓ_∞^n has maximal volume ratio: (this fact was used in the proof of Theorem 2.5.2 above). The inequality of Brascamp and Lieb can be applied equally well to estimate the volume ratios of subspaces of L_p for $1 \leq p < \infty$, since the volumes of their unit balls can be easily expressed as convolutions: (this is done in Lemma 7 which was observed in [45]).

Theorem 2.5.7. *Let $1 \leq p \leq \infty$. Among n-dimensional subspaces of L_p, the space ℓ_p^n has exactly maximal volume ratio.*

The remainder of this section is devoted to the proof of Theorem 2.5.7.

Lemma 2.5.8. *Let K be a symmetric convex body in $\mathbb{R}^n$ with Minkowski gauge $\|\cdot\|$ and $1 \leq p < \infty$. Then*

$$|K| = \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} e^{-\|x\|^p}dx.$$

Proof. Let σ be the rotation invariant probability on the Euclidean sphere S^{n-1} and v_n the volume of the Euclidean unit ball B_2^n. Then

$$\begin{aligned}&\int_{\mathbb{R}^n} e^{-\|x\|^p}dx = nv_n\int_{S^{n-1}}\int_0^\infty e^{-\|r\theta\|^p}r^{n-1}drd\sigma(\theta)\\ &= nv_n\int_{S^{n-1}}\|\theta\|^{-n}d\sigma(\theta)\cdot\int_0^\infty e^{-r^p}r^{n-1}dr\\ &= n|K|\int_0^\infty e^{-r^p}r^{n-1}dr\\ &= \Gamma\Big(1+\frac{n}{p}\Big)|K|.\end{aligned}$$

□

Proposition 2.5.9. *Let $(u_i)_1^m$ be unit vectors in $\mathbb{R}^n$ and $(c_i)_1^m$ positive numbers satisfying*

$$\sum_1^m c_i u_i \otimes u_i = I_n,$$

$(\alpha_i)_1^m$ positive numbers and $1 \leq p < \infty$. For each $x \in \mathbb{R}^n$ set

$$\|x\| = \Big(\sum_1^m \alpha_i|\langle u_i, x\rangle|^p\Big)^{\frac{1}{p}}.$$

Then if K is the unit ball of the space $(\mathbb{R}^n, \|\cdot\|)$,

$$|K| \leq \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\prod_{i=1}^m\Big(\frac{c_i}{\alpha_i}\Big)^{\frac{c_i}{p}}.$$

Remark 8. *For $1 \leq p \leq 2$. Meyer and Pajor, [M-P], proved that the largest n-dimensional sections of the unit ball of ℓ_p^m are those spanned by n standard unit vectors. This result is a consequence of Proposition 8 applied with $\alpha_i = c_i^{\frac{p}{2}}, 1 \leq i \leq m$.*

Proof. For $1 \leq i \leq m$, let $f_i\colon\ \mathbb{R} \to [0,\infty)$ be defined by

$$f_i(t) = \exp\Big(-\frac{\alpha_i}{c_i}|t|^p\Big).$$

By Lemmas 2.5.5 and **??**,

$$
\begin{aligned}
|K| &= \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} \exp\Big(-\sum \alpha_i |\langle u_i, x\rangle|^p\Big)\,dx \\
&= \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} \prod_{i=1}^{n}\Big(\exp\frac{-\alpha_i}{c_i}|\langle u_i, x\rangle|^p\Big)^{c_i} dx \\
&= \frac{1}{\Gamma(1+\frac{n}{p})}\int_{\mathbb{R}^n} \prod_{i=1}^{m} f_i(\langle u_i, x\rangle)^{c_i}\,dx \\
&\le \frac{1}{\Gamma(1+\frac{n}{p})}\prod_{i=1}^{m}\Big(\int_{\mathbb{R}} f_i\Big)^{c_i} \\
&= \frac{1}{\Gamma(1+\frac{n}{p})}\prod_{i=1}^{m}\Big(2\Big(\frac{c_i}{\alpha_i}\Big)^{\frac{1}{p}}\Gamma\Big(1+\frac{1}{p}\Big)\Big)^{c_i} \\
&= \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}\prod_{i=1}^{m}\Big(\frac{c_i}{\alpha_i}\Big)^{\frac{c_i}{p}}.
\end{aligned}
$$

The deduction of Theorem 2.5.7 from the preceding proposition uses an important lemma of Lewis [46], which extends John's theorem, Lemma 2.5.3.

□

Lemma 2.5.10. *Let $1 \le p < \infty$ and X be an n-dimensional subspace of ℓ_p^m. Then X may be represented on $\mathbb{R}^n$ with norm given by*

$$\|x\| = \Big(\sum_{1}^{m} c_i |\langle u_i, x\rangle|^p\Big)^{\frac{1}{p}}$$

where $(u_i)_1^m$ is a sequence of unit vectors and $(c_i)_1^m$ a sequence of positive numbers satisfying

$$\sum_{1}^{m} c_i u_i \otimes u_i = I_n.$$

Remark 9. *The key point in Lemma 9 is that the same c_i's appear in both expressions. Thus, Proposition 8 may be applied with $\alpha_i = c_i, 1 \le i \le m$; in this case the estimate of Proposition 8 has a particularly simple form.*

proof of theorem 2.5.7. It may be assumed that $p < \infty$ and it suffices to estimate the volume ratios of n-dimensional subspaces of ℓ_p^m for each integer $m \ge n$. Let X be such a space and assume that it is represented on $\mathbb{R}^n$ in the way guaranteed by Lemma 9, with unit ball K (say). By Proposition 8,

$$|K| \le \frac{2^n\Gamma(1+\frac{1}{p})^n}{\Gamma(1+\frac{n}{p})}.$$

The latter expression is the volume of the unit ball of ℓ_p^n in its normal representation on $\mathbb{R}^n$. So it is enough to check that K contains an Euclidean ball of radius

$$\begin{cases} n^{\frac{1}{2}-\frac{1}{p}} & \text{if} \quad 1 \le p \le 2 \\ 1 & \text{if} \quad p > 2 \end{cases} .$$

In the first case, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \|x\|^p &= \sum_1^m c_i |\langle u_i, x\rangle|^p \\ &\le \Big(\sum_1^m c_i\Big)^{1-\frac{p}{2}} \Big(\sum_1^m c_i \langle u_i, x\rangle^2\Big)^{\frac{p}{2}} \\ &= n^{1-\frac{p}{2}} |x|^p \end{aligned}$$

($|x|$ being the Euclidean length of x). In the second case,

$$\|x\|^p = \sum_1^m c_i |\langle u_i, x\rangle|^p \le \sum_1^m c_i |x|^{p-2} \langle u_i, x\rangle^2 = |x|^{p-2} \cdot |x|^2 = |x|^p.$$

□

2.5.4 Appendix.

For a convex body C in $\mathbb{R}^n$ and unit vector $\theta \in \mathbb{R}^n$, let $P_\theta C$ be the orthogonal projection of C onto the 1-codimensional subspace of $\mathbb{R}^n$ perpendicular to θ. It was observed by Petty in [47], that the expression

$$\left(|C|^{n-1} \int_{S^{n-1}} |P_\theta C|^{-n} d\sigma(\theta) \right)^{-\frac{1}{n}} \tag{5}$$

is invariant under invertible affine transformations of the body C. This expression measures "minimal surface area" in the sense of Theorems 1 and 2. To see this, note that, on the one hand, the Cauchy formula for surface area states that for a convex body C in $\mathbb{R}^n$,

$$|\partial C| = \frac{n v_n}{v_{n-1}} \int_{S^{n-1}} |P_\theta C| d\sigma(\theta)$$

and so by Hölder's inequality

$$\frac{|\partial C|}{|C|^{\frac{n-1}{n}}} \ge \frac{n v_n}{v_{n-1}} \left(|C|^{n-1} \int_{S^{n-1}} |P_\theta C|^{-n} d\sigma(\theta) \right)^{-\frac{1}{n}}.$$

On the other hand, Theorem 2.5.7 (for $p = 1$) states that if X is an n-dimensional subspace of L_1 then

$$vr(X) \le \Big(\frac{2^n \Gamma(1 + \frac{n}{2})}{\Gamma(1+n) \pi^{\frac{n}{2}}} \Big)^{\frac{1}{n}} \le \sqrt{\frac{2e}{\pi}}.$$

From this it is easy to deduce that each convex body C has an affine image $\widetilde{C}$ for which

$$\begin{aligned}\frac{|\partial\widetilde{C}|}{|\widetilde{C}|^{\frac{n-1}{n}}} &\leq \sqrt{\frac{2e}{\pi}}\frac{nv_n}{v_{n-1}}\left(|\widetilde{C}|^{n-1}\int_{S^{n-1}}|P_\theta\widetilde{C}|^{-n}d\sigma(\theta)\right)^{-\frac{1}{n}}\\ &= \sqrt{\frac{2e}{\pi}}\frac{nv_n}{v_{n-1}}\Big(|C|^{n-1}\int_{S^{n-1}}|P_\theta C|^{-n}d\sigma(\theta))^{-\frac{1}{n}}.\end{aligned}$$

A strong isoperimetric inequality of Petty, [48], states that the expression (5) is minimised by the Euclidean balls. It would be possible to reverse the isoperimetric inequality by determining the bodies which maximise (5). It seems likely that the cube and tetrahedron are extremal for this modified problem: this is certainly true if $n = 2$.

2.6 On the volume of the intersection of two L_p^n balls

G. Schechtman[‡] and J. Zinn[§] [¶]

2.6.1 Introduction.

This note deals with the following problem, the case $p = 1$, $q = 2$ of which was introduced to us by Vitali Milman: What is the volume left in the L_p^n ball after removing a t-multiple of the L_q^n ball? Recall that the L_r^n ball is the set $\{(t_1, t_2, \ldots, t_n);\ t_i \in \mathbf{R},\ n^{-1}\sum_{i=1}^n |t_i|^r \le 1\}$ and note that for $0 < p < q < \infty$ the L_q^n ball is contained in the L_p^n ball.

In Corollary 4 below we show that, after normalizing Lebesgue measure so that the volume of the L_p^n ball is one, the answer to the problem above is of order $e^{-ct^p n^{p/q}}$ for $T < t < \frac{1}{2}n^{\frac{1}{p}-\frac{1}{q}}$, where c and T depend on p and q but not on n.

The main theorem, Theorem 3, deals with the corresponding question for the surface measure of the L_p^n sphere. Theorem 3 and Corollary 4 together with some other remarks form Section 3. In Section 2 we introduce a class of random variables to be used in the proof of the main theorem. These random variables are related to L_p in the same way that Gaussian variables are related to L_2.

2.6.2 Preliminaries.

Here we introduce a class of random variables to be used in the proof of the main theorem and summarize some of their properties. Fix a $0 < p < \infty$ and let $x, x_1, x_2, \ldots, x_n$ be independent random variables each with density function $c_p e^{-t^p}$, $t > 0$. Note that necessarily $c_p = p/\Gamma(1/p)$. The first claim is known, though we could not locate a reference.

Lemma 2.6.1. *Put* $S = \left(\sum_{i=1}^n x_i^p\right)^{1/p}$*, then* $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ *is uniformly distributed over the positive quadrant of the sphere of* l_p^n*, i.e., over the set* $\Delta_p = \{(t_1, t_2, \ldots, t_n)\ ;\ t_i \ge 0,\ \sum t_i^p = 1\}$ *equipped with the* $(n-1)$*-dimensional normalized Lebesgue measure. Moreover,* $\left(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\right)$ *is independent of* S*.*

Proof. For any Borel subset A of Δ_p,

$$\begin{aligned}
&P\Big(\Big(\frac{x_1}{S}, \frac{x_2}{S}, \ldots, \frac{x_n}{S}\Big) \in A \Big| S = a\Big) = \\
&= \lim_{\epsilon\to 0} \frac{P((x_1, \ldots, x_n) \in \mathbf{R}_+ A \ \&\ a-\epsilon \le S \le a+\epsilon)}{P(a-\epsilon \le S \le a+\epsilon)} \\
&= \lim_{\epsilon\to 0} \int_{\substack{(t_1,\ldots,t_n)\in\mathbf{R}_+ A \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} e^{-\sum t_i^p} dt \Big/ \int_{\substack{(t_1,\ldots,t_n)\in\mathbf{R}_+^n \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} e^{-\sum t_i^p} dt \\
&\le \limsup_{\epsilon\to 0} e^{-(a-\epsilon)^p + (a-\epsilon)^p}\Big)^p \int_{\substack{(t_1,\ldots,t_n)\in\mathbf{R}_+ A \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} dt \Big/ \int_{\substack{(t_1,\ldots,t_n)\in\mathbf{R}_+^n \\ (a-\epsilon)^p < \sum t_i^p < (a+\epsilon)^p}} dt \\
&= \lambda(A),
\end{aligned}$$

[‡] Supported in part by the US- Israel BSF and by the Glikson Foundation

[§] Supported in part by NSF DMS-86-01250 and by Texas Advanced Research Program

[¶] Grant no. 3825

where λ is the normalized Lebesgue measure on Δ. Similarly,

$$P\Big(\Big(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S}\Big) \in A \Big| S = a\Big) \geq \lambda(A).$$

This proves that $P\Big(\Big(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S}\Big) \in A\Big) = \lambda(A)$ and that $\Big(\frac{x_1}{S}, \frac{x_2}{S}, \dots, \frac{x_n}{S}\Big)$ is independent of S. ☐

In the next claim we gather some more properties of the random variables x_i.

Lemma 2.6.2. *Let $x, x_1, \dots, x_n$ be as above, then*

1. c_p is bounded away from zero and infinity when $p \to \infty$.

2. For all $h > 0$ and all $0 < p < \infty$, $\mathbf{E}e^{-hx^p} = \Big(\frac{1}{1+h}\Big)^{1/p}$. In particular,

$$\mathbf{E}e^{-hx^p} \geq e^{-h/p} \quad for\ all \quad h > 0 \quad and \quad \mathbf{E}e^{-hx^p} \leq e^{-h/2p} \quad for\ all \quad 0 < h \leq 1.$$

3. For all $0 < u < \infty$ and all $0 < p < \infty$, $P(x^p > u) \geq \frac{c_p}{2p}e^{-2u}$. If $p \geq 1$ and $u \geq 1$, then also $P(x^p > u) \leq \frac{c_p}{p}e^{-u/2}$. In particular, for $p \geq 1$ and all u, $P(x^p > u) \leq Ce^{-u/2}$ for some universal C.

4. For all $1 \leq p \leq q < \infty$, $\mathbf{E}\Big(\sum_{i=1}^n x_i^q\Big)^{1/q}$ is equivalent, with universal constants, to $q^{1/p}n^{1/q}$, if $q \leq \log n$, and to $(\log n)^{1/p}$ otherwise.

Proof. 1. Follows easily from the fact that $c_p = p/\Gamma(1/p) = \Gamma(\frac{1}{p}+1)^{-1}$.

2. is a simple computation.

3. is also simple, here is a sketch of the proof.

$$\begin{aligned} P(x^p > u) &= c_p \int_{u^{1/p}}^{\infty} e^{-t^p} dt \\ &\geq c_p \int_{u^{1/p}}^{(u+1)^{1/p}} \frac{pt^{(p-1)}}{p(u+1)^{(p-1)/p}} e^{-t^p} dt \\ &= \frac{c_p}{p(u+1)^{(p-1)/p}} \Big(1 - \frac{1}{e}\Big) e^{-u} \\ &\geq \frac{c_p}{2p(u+1)} e^{-u} \\ &\geq \frac{c_p}{2p} e^{-2u}. \end{aligned}$$

The other inequality in 3 is proved in a similar way.

4. First note that for all $0 < p, q < \infty$

$$\mathbf{E}x^q = c_p \int_0^{\infty} t^q e^{-t^p} dt = \frac{c_p}{p}\Gamma\Big(\frac{q+1}{p}\Big)$$

so that, by the triangle inequality and 1, if $1 \le p \le q < \infty$

$$\mathbf{E}\Big(\sum_{i=1}^{n} x_i^q\Big)^{1/q} \le \Big(\sum_{i=1}^{n} \mathbf{E}x_i^q\Big)^{1/q} = \Big(\frac{c_p}{p}\Gamma\Big(\frac{q+1}{p}\Big)\Big)^{1/q} n^{1/q} \le Cq^{1/p}n^{1/q}$$

for some universal C. For the lower bound in the case $q \le \log n$, divide $\{1, 2, \dots, n\}$ into approximately n/e^q disjoint sets of cardinality approximately e^q each, then

$$\begin{aligned}
&\mathbf{E}\Big(\sum_{i=1}^{n} x_i^q\Big)^{1/q} = \mathbf{E}\Big(\sum_{j}\Big(\sum_{i\in\sigma_j} x_i^q\Big)^{q/q}\Big)^{1/q} \\
&\ge \mathbf{E}\Big(\sum_{j}\big(\max_{i\in\sigma_j} x_i\big)^q\Big)^{1/q} \\
&\ge \Big(\sum_{j}\big(\mathbf{E}\max_{i\in\sigma_j} x_i\big)^q\Big)^{1/q} \\
&\ge c'(\log e^q)^{1/p}(n/e^q)^{1/q} \\
&\ge c'' q^{1/p} n^{1/q}.
\end{aligned}$$

Now, for the case $q > \log n$ we note first that, by 3,

$$P(\max_{1\le i\le n} x_i > t) \ge 1 - \big(1 - \frac{c_p}{2p}e^{-2t^p}\big)^n.$$

For n smaller than an absolute multiple of p, the lower bound follows easily from the fact that $\mathbf{E}x_1$ is larger that a universal positive constant, so assume that $n \ge 20p/c_p$ and put $t = 2^{-1/p}\Big(\log \frac{nc_p}{2p}\Big)^{1/p}$.Then, for some universal c,

$$P(\max_{1\le i\le n} x_i > c(\log n)^{1/p}) \ge 1/2.$$

In particular, $\mathbf{E}\max_{1\le i\le n} x_i \ge c(\log n)^{1/p}$, which implies the lower bound in this case since $\Big(\sum_{i=1}^{n} x_i^q\Big)^{1/q}$ is universally equivalent to $\max_{1\le i\le n} x_i$. The upper bound in this case, though a bit harder, is also standard and since we don't use it in the sequel we shall leave it to the reader. □

The statement in 4, for the case $p = 2$, was noticed by the first named author several years ago while seeking a precise estimate for the dimension of the Euclidean sections of l_p^n spaces (see [49] p.145 Remark 5.7). The original proof was more complicated. The proof presented here is an adaptation of a proof of the case $p = 2$ shown to us by J. Bourgain.

2.6.3 The main result

Theorem 2.6.3. *For all $1 \le p < q < \infty$ there are constants $c = c(p, q)$, $C = C(p, q)$ and $T = T(p, q)$ such that if μ denotes the normalized Lebesgue measure on the positive quadrant of the unit sphere of L_p^n then*

$$\mu(\|u\|_{L_q^n} > t) \le \exp(-ct^p n^{p/q}) \qquad (1)$$

for all $t > T$*, and*

$$\mu(\|u\|_{L_q^n} > t) \geq \exp(-Ct^p n^{p/q}) \tag{2}$$

for all $2 \leq t \leq \frac{1}{2} n^{\frac{1}{p}-\frac{1}{q}}$.

Moreover, for $q > 2p$ *(or any other universal positive multiple of* p*), one can take* $c(p,q) = \frac{\gamma}{p}$*,* $C(p,q) = \frac{\Gamma}{p}$ *and* $T(p,q) = \tau \min\{q, logn\}^{1/p} \leq q^{1/p}$*. Here* γ *,*Γ *and* τ *are universal constants.*

Proof. By Lemma 1 above,

$$\mu(\|u\|_{L_q^n} > t) = P(n^{\frac{1}{p}-\frac{1}{q}}(\sum_{i=1}^n x_i^q)^{1/q}/(\sum_{i=1}^n x_i^p)^{1/p} > t)$$

where x_i are independent random variables each with density $c_p e^{-t^p}$. Assume, for the simplicity of the presentation, that n is even. Put $S = (\sum_{i=1}^n x_i^p)^{1/p}$ and let $p_j, \quad j = 1, 2, \ldots, n/2$ be positive numbers with sum $\leq 1/2$. Then

$$\begin{aligned}
&P\big(n^{\frac{1}{p}-\frac{1}{q}}(\sum_{i=1}^n x_i^q)^{1/q}/(\sum_{i=1}^n x_i^p)^{1/p} > t\big) = \\
&= P\Big(\sum_{i=1}^n x_i^q > \frac{t^q(\sum_{i=1}^n x_i^p)^{q/p}}{n^{\frac{q}{p}-1}}\Big) \\
&\leq \sum_{i=1}^{n/2} P(x_i^* > tp_i^{1/q} S/n^{\frac{1}{p}-\frac{1}{q}}) + P\Big(\sum_{i=\frac{n}{2}+1}^{n} x_i^{*q} > t^q S^q/2n^{\frac{q}{p}-1}\Big) \qquad (3)
\end{aligned}$$

where $\{x_j^*\}$ denotes the nonincreasing rearrangement of $\{|x_j|\}$.

Since

$$\begin{aligned}
\sum_{j=\frac{n}{2}+1}^{n} x_i^{*q} &\leq \frac{n}{2} x_{\frac{n}{2}}^{*q} \leq \frac{n}{2}(\frac{2}{n}\sum_{i=1}^{n/2} x_i^{*p})^{q/p} \\
&\leq 2^{\frac{q}{p}-1} S^q/n^{\frac{q}{p}-1} \quad ,
\end{aligned}$$

we get that, if $t \geq 2^{1/p}$, the second term in (3) is zero.

To evaluate the first term in (3), fix $1 \leq j \leq n/2$. Then,

$$\begin{aligned}
P(x_j^* > tp_j^{1/q} S/n^{\frac{1}{p}-\frac{1}{q}}) &\leq \binom{n}{j} P(x_1, \ldots, x_j > tp_j^{1/q} S/n^{\frac{1}{p}-\frac{1}{q}}) \\
&\leq \binom{n}{j} P\big(x_1^p, \ldots, x_j^p > t^p p_j^{p/q} \sum_{i=j+1}^n x_i^p/n^{1-p/q}\big).
\end{aligned}$$

From Lemma 2 (first 3 and then 2) we get that the last expression is dominated by

$$\binom{n}{j} C^j \mathbf{E} \exp\big(-jp_j^{p/q} t^p \sum_{i=j+1}^n x_i^p/n^{1-p/q}\big)$$

$$\leq \binom{n}{j} C^j \exp(-jp_j^{p/q} t^p (n-j)/2pn^{1-p/q})$$

for some universal C. Note that the last inequality holds if $jn^{p/q-1}p_j^{p/q}t^p \le 1$. If this is not the case the probability we are trying to evaluate is zero. Finally, the last term is dominated by

$$\exp\Big(j\big(log\frac{en}{j} + C - \frac{p_j^{p/q}t^pn^{p/q}}{4p}\big)\Big). \tag{4}$$

Now, for α to be chosen momentarily, let p_j, $j = 1, \dots, n/2$, be such that

$$j\big(log\frac{en}{j} + C - \frac{p_j^{p/q}t^pn^{p/q}}{4p}\big) = -\alpha n^{p/q}t^p$$

i.e.,

$$p_j = \big(4p\frac{log\frac{en}{j}}{t^pn^{p/q}} + \frac{4Cp}{t^pn^{p/q}} + \alpha\frac{4p}{j}\big)^{q/p}.$$

We thus get that,for some universal constant C,

$$p_j \le 2^{\frac{q}{p}-1}\frac{(Cp)^{q/p}(log\frac{en}{j})^{q/p}}{t^qn} + 2^{\frac{q}{p}-1}\alpha^{q/p}\frac{(4p)^{q/p}}{j^{q/p}}. \tag{5}$$

It is easy to see that, for $1 \le p < q < \infty$,

$$\sum_{j=1}^{n/2}(log\frac{en}{j})^{q/p} \le An\min\{q^{q/p}, (logn)^{q/p}\}$$

for some universal A. Thus the sum over j of the first terms in (5) is smaller than $1/4$ if, for some universal γ, $t > \gamma\min\{q^{1/p}, (\log n)^{1/p}\}$. The sum over j of the second terms in (5) is bounded by $1/4$ if $\alpha < B\frac{1}{p}(\frac{q}{p} - 1)^{p/q}$, for some universal B. Choosing α to satisfy this inequality and using (3),(4) and (5) we get that, for $t > \gamma\min\{q^{1/p}, (logn)^{1/p}\}$,

$$\mu(\|u\|_{L_q^n} > t) \le \frac{n}{2}e^{-\alpha n^{p/q}t^p}.$$

Under the conditions on t, the factor $n/2$ can be absorbed in the second term (changing α to another constant of the same order of magnitude as a function of p), thus proving (1).

We now turn to the proof of the lower bound (2) which is simpler. Using Claim 1 again,

$$\begin{aligned}
&\mu\|u\|_{L_q^n} > t) = P(n^{\frac{1}{p}-\frac{1}{q}}(\sum_{i=1}^n x_i^q)^{1/q}/(\sum_{i=1}^n x_i^p)^{1/p} > t)\\
&\ge P(x_1 > St/n^{\frac{1}{p}-\frac{1}{q}})\\
&= P\Big(x_1 > \frac{t}{(n^{(1-p/q)} - t^p)^{1/p}}\big(\sum_{i=2}^n x_i^p\big)^{1/p}\Big).
\end{aligned}$$

Since $t^p \le \frac{1}{2}n^{(1-p/q)}$, this dominates

$$P\Big(x_1 > \frac{2^{1/p}t}{n^{\frac{1}{p}-\frac{1}{q}}}\big(\sum_{i=2}^n x_i^p\big)^{1/p}\Big).$$

Now, by Claim 2.3.,

$$P\Big(x_1 > \frac{2^{1/p}t}{n^{\frac{1}{p}-\frac{1}{q}}}\big(\sum_{i=2}^n x_i^p\big)^{1/p}\Big) \ge \frac{c_p}{2p}\mathbf{E}\exp(-4t^p\sum_{i=2}^n x_i^p/n^{(1-p/q)})$$

$$
\begin{aligned}
&= \frac{c_p}{2p}\big(\mathbf{E}\exp(-4t^p x_1^p/n^{(1-p/q)})\big)^{n-1} \\
&= \frac{c_p}{2p}\Big(\frac{1}{1+\frac{4t^p}{n^{1-p/q}}}\Big)^{(n-1)/p} && \textit{(by Claim 2.2.)} \\
&\geq \frac{c_p}{2p}\exp\Big(-\frac{4t^p(n-1)}{pn^{(1-p/q)}}\Big) \\
&\geq \frac{c_p}{2p}e^{4t^p n^{p/q}/p}.
\end{aligned}
$$

Finally observe that, since c_p is bounded away from zero and $t \geq 2$, the factor $\frac{c_p}{2p}$ can be absorbed in the second term (changing 4 to another universal constant).

□

Remark 10. *1. It follows from the proof that, for n large enough and q close to p, one can take* $c(p,q) = \frac{c}{p}\Big(\frac{q}{p}-1\Big)$ *for some universal constant c.*

2. It follows from the statement of the theorem that, for $q=\infty$*,*

$$\mu(\|u\|_\infty > t) \leq e^{-\gamma t^p/p}$$

for all $t > \tau(\log n)^{1/p}$*, and*

$$\mu(\|u\|_\infty > t) \geq e^{-\Gamma t^p/p}$$

for all $2 \leq t \leq \frac{1}{2}n^{\frac{1}{p}}$*, where* γ*,* Γ *and* τ *are universal constants.*

3. Note that it follows from Claim 1 and Claim 2.4. that the order of magnitude of T is the correct one.

4. The restriction $p \geq 1$ *in Theorem 3 above and in Corollary 4 below can be replaced by* $p > 0$ *if one replaces the inequality* $t \geq 2$ *with* $t \geq d$*, for some d depending only on p and q, and removes the "moreover" part. We didn't check the dependence of the constants on p and q in this case.*

The last remark is that one can get a similar statement for the full balls. We state it as a corollary.

Corollary 2.6.4. *For all* $1 \leq p < q < \infty$ *there are constants* $c = c(p,q)$ *,* $C = C(p,q)$ *and* $T = T(p,q)$ *such that if* ν *denotes the normalized Lebesgue measure on the ball of* L_p^n *then, for all n large enough,*

$$\nu(\|u\|_{L_q^n} > t) \leq \exp(-ct^p n^{p/q}) \tag{6}$$

for all $t > T$*, and*

$$\nu(\|u\|_{L_q^n} > t) \geq \exp(-Ct^p n^{p/q}) \tag{7}$$

for all $2 \leq t \leq \frac{1}{2}n^{\frac{1}{p}-\frac{1}{q}}$*. Moreover, for* $q > 2p$ *(or any other universal positive multiple of p), one can take* $c(p,q) = \frac{\gamma}{p}$ *,* $C(p,q) = \frac{\Gamma}{p}$ *and* $T(p,q) = \tau\min\{q, logn\}^{1/p} \leq q^{1/p}$*, where* γ *,*Γ *and* τ *are universal constants.*

The proof follows easily from Theorem 3 and the formula

$$\nu(A) = n\int_0^1 r^{n-1}\mu\Big(\frac{A}{r}\Big)dr$$

which holds for all Borel sets A in the ball of L_p^n.

2.7 The Rademacher Cotype of Operators from l_∞^N

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Abstract:
We show that for any operator: $T : l_\infty^N \to Y$, where Y is a Banach space, that its cotype 2 constant,$K^{(2)}(T)$ is related to its $(2,1)$-summing norm, $\pi_{2,1}(T)$, by $K^{(2)}(T) \leq c.loglogN\pi_{2,1}(T)$. Thus, we can show that there is an operator $T : C(K) \to Y$ that has cotype 2, but is not 2-summing.

A.M.S. Classification: Primary 46B20, Secondary 60G99

2.7.1 Introduction.

The notation we use in this paper is loosely based on that given in[50–52].
we let $\epsilon_1, \epsilon_2, \ldots$ be independent Rademacher random variables, that is, $Pr(\epsilon_s = 1) = Pr(\epsilon_s = -1) = \frac{1}{2}$ A linear operator $T : X \to Y$ is said to have (Rademacher) cotype $p (p \geq 2)$ if there is a constant $C < \infty$ such that for all $x_1, x_2, \ldots, x_S$ in X we have

$$\left(\sum_{s=1}^{s} \|T(x_s)\|^p \right)^{\frac{1}{p}} \leq c.\mathbb{E}\| \sum_{s=1}^{s} \epsilon_s . x_s \|.$$

The smallest value of C is called the (Rademacher) cotype p constant of T, and is denoted by $K^{(p)}(T)$. These definitions extend to spaces in the obvious way; a space X has cotype p if its identity operator has cotype p.
We define the (p,q)-summing norm of a linear operator $T : X \to Y$, denoted by $\pi_{p,q}(T)$, to be the least number C such that for all $x_1, x_2, \ldots, x_S$ in X we have

$$\left(\sum_{s=1}^{S} \|T\left(x_s\right)\|^p \right)^{\frac{1}{p}} \leq C \sup \left\{ \left(\sum_{s=1}^{S} |\langle x^*, x_s \rangle|^q \right)^{\frac{1}{q}} \right\}$$

where the supremum is taken over all x^* in the unit ball of X^*. We call a (p,p)-summing operator a p-summing operator, and write $\pi_p(T)$ for $\pi_{p,p}(T)$. We say that the operator is (p,q)-summing (p-summing) if $\pi_{p,q}(T) < \infty$ (respectively $\pi_p(T) < \infty$).

If $1 \leq p < \infty$, and $1 \leq q \leq \infty$, then we let $L_{p,q}(\mu)$ denote the Lorentz space on the measure μ. We refer the reader to [51, 53] for details, but just note that the $L_{p,1}$ norm may be calculated using

$$\|f\|_{p,1} = \int_0^\infty \mu(|f| > t)^{\frac{1}{p}} dt = \frac{1}{p} \int_0^\infty s^{\frac{1}{p}-1} f^*(s) ds$$

where f^* denotes the non-decreasing rearrangement of $|f|$.

The basic motivation behind this paper is in classifying operators from $C(K)$ that factor through a Hilbert space, where $C(K)$ denotes the continuous functions on the compact Hausdorff topological space, K. The first result in this direction is due to Grothendieck, which states that any bounded linear operator $C(K) \to L_1$ factors through Hilbert space. This was generalized by Maurey [54], allowing L_1 to be replaced by any space of cotype 2, to give the following result (see also [52]).

Theorem 2.7.1. *Let $T : C(K) \to Y$ be a linear operator, where Y is any Banach space. Then the following are equivalent:*
i) T is 2-summing;

ii) T factors through Hilbert space;

iii) T factors through a space of cotype 2.

However, we are still left with the following question: if the operator $T : C(K) \to Y$ has cotype 2 , does it follow that it factors through Hilbert space?

One way one might tackle this problem is to consider the $(2, 1)$-summing norms of such operators. Jameson [55] showed that there is an operator $T : l_\infty^N \to Y$ such that $\pi_2(T) \geq c^{-1}\sqrt{\log N}\pi_{2,1}(T)$. Hence, if we can establish a strong relationship between the cotype 2 constants and the $(2, 1)$-summing norms of such operators, then we can answer the above question in the negative. To this end, we have the following - the main result of this paper.

Theorem 2.7.2. *There is a constant c such that for any operator $T : l_\infty^N \to Y$, where Y is a Banach space, then the cotype 2 constant is bounded according to the relation:*

$$K^{(2)}(T) \leq c \log\log N \pi_{2,1}(T)$$

Corollary 2.7.3. *There is an operator $T : C(K) \to Y$, where Y is a Banach space, that has cotype 2, but does not factor through Hilbert space.*

Finally, before embarking on the proof of this result, we point out that for $p > 2$, the above problems have been completely answered.

Theorem 2.7.4. *Let $T : C(K) \to Y$ be a bounded linear operator, where Y is a Banach space. Then for all $p > 2$, the following are equivalent:*
i) T is ($p, 1$)-summing;

ii) T has Rademacher cotype p;

iii) T factors through a space with Rademacher cotype p.

The implication (i) ⇔ (ii) is due to Maurey [56]. The third equivalence follows from the fact that any $(p, 1)$-summing operator from $C(K)$ factors through $L_{p,1}$ (see [57] or Theorem 5 below), and that $L_{p,1}$ has Rademacher cotype p, (see [58]).

Theorem 2.7.5. *If $p > 2$, then there is a bounded linear operator $C(K) \to L_p$ that is not -summing.*

We refer the reader to [59].

2.7.2 Proof of the Main Result

To prove Theorem 2, we need the following two results. The first allows us to reduce questions about $(p,1)$-summing operators from $C(K)$ to the canonical embedding $C(K) \to L_{2,1}(K,\mu)$ (μ a probability measure), and is due to Pisier (see [57]).

Theorem 2.7.6. *Let* $T : C(K) \to Y$ *be a* $(p,1)$*-summing operator, where* Y *is a Banach space, and* $p \geq 1$. *Then there is a Radon probability measure* μ *on* K *and a constant* $C \leq p^{\frac{1}{p}} \pi_{p,1}(T)$ *such that for all* $x \in C(K)$ *we have* $\|Tx\| \leq C\|x\|_{L_{p,1}(K,\mu)}$.

The second result is about Rademacher processes, and is due to the second named author (for the proof, see [60]). First we establish some more notation. If T is a bounded subset of $\mathbb{R}^S$, we write

$$r(T) = \mathbb{E} \sup_{t \in T} \left| \sum_{s=1}^{S} \varepsilon_s t(s) \right|$$

If B is a subset of $\mathbb{R}^S$, we write $\mathcal{N}(T,B)$ for the minimal number of translates of B required to cover D. We write B_1^S for the unit ball of l_1^S, and B_2^S for the unit ball of l_2^S. From now on, we take all logarithms to base 2 .

Theorem 2.7.7. *There is a constant* c_1 *such that if* T *is a bounded subset of* $\mathbb{R}^S$, *and* $\epsilon > 0$, *then letting* $D = c_1 r(T) B_1^S + \epsilon B_2^S$, *we have*

$$r(T) \geq c_1^{-1} \epsilon \sqrt{\log \mathcal{N}(T,D)}$$

Now we will state the main result towards proving Theorem 2.

Proposition 2.7.8. *There is a constant* c_2 *such that if* $(\Omega, \mathcal{F}, \mu)$ *is a probability space with* N *atoms, and* $x_1, x_2, \ldots, x_S \in L_\infty(\mu)$ *are such that*

$$\mathbb{E} \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\|_\infty \leq 1$$

then

$$\left(\sum_{s=1}^{S} \|x_s\|_{L_{2,1}(\mu)}^2 \right)^{\frac{1}{2}} \leq c_2 \log \log N$$

Our first step in establishing this result is to restate Theorem 6 in a more suitable form.

Lemma 2.7.9. *There is a constant* c_1 *(the same one as in Theorem 6) such that the following holds. Suppose that* $(\Omega, \mathcal{F}, \mu)$ *is a measure space, with* Ω *finite, and* $x_1, x_2, \ldots, x_S \in L_\infty(\mu)$ *with*

$$\mathbb{E} \left\| \sum_{s=1}^{S} \varepsilon_s x_s \right\|_\infty \leq 1$$

Then for all integers k, we may partition Ω into at most 2^{2^k} measurable sets, find $y_1, y_2, \ldots, y_S, z_1, z_2, \ldots, z_S \in L_\infty(\mu)$, and find $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_S \in L_\infty(\Omega, \mathcal{F}', \mu)$ (where $\mathcal{F}'$ denotes the algebra generated by the partition), such that $x_s = \bar{x}_s + y_s + z_s$,

$$\mathbb{E}\left\|\sum_{s=1}^S \varepsilon_s \bar{x}_s\right\|_\infty \leq 1, \quad \left\|\sum_{s=1}^S |y_s|\right\|_\infty \leq c_1 \quad \text{and} \quad \left\|\left(\sum_{s=1}^S |z_s|^2\right)^{\frac{1}{2}}\right\|_\infty \leq c_1 2^{-\frac{k}{2}}$$

Proof. Let $T = \left\{(x_s(\omega))_{s=1}^S : \omega \in \Omega\right\}$, and let $\epsilon = c_1^{-1} 2^{-\frac{k}{2}}$. If we apply Theorem 6 , we see that there are 2^{2^k} translates, $t_l + c_1\left(B_1^S + 2^{-\frac{k}{2}} B_2^S\right)\left(1 \leq l \leq 2^{2^k}\right)$, that cover T. We let the covering of Ω be the sets

$$A_l = \left\{\omega : (x_s(\omega))_{s=1}^S \in t_l + c_1\left(B_1^S + 2^{-\frac{k}{2}} B_2^S\right)\right\}$$

and if A_l is non-empty, we choose $\omega_l \in A_l$. Define $\bar{x}_s(\omega) = x_s(\omega_l)$ if $\omega \in A_l$. Now, if $\omega \in A_l$, we know that $(x_s(\omega) - \bar{x}_s(\omega))_{s=1}^S \in c_1\left(B_1^S + 2^{-\frac{k}{2}} B_2^S\right)$, that is, there are $(y_s(\omega))_{s=1}^S \in c_1 B_1^S$ and $(z_s(\omega))_{s=1}^S \in c_1 2^{-\frac{k}{2}} B_2^S$, with $x_s(\omega) = \bar{x}_s(\omega) + y_s(\omega) + z_s(\omega)$. □

Lemma 2.7.10. *There is a constant c_3 such that if $(\Omega, \mathcal{F}, \mu)$ is a measure space with Ω finite, then the following hold.*

i) If $y \in L_\infty(\mu)$, then $\|y\|_{2,1} \leq \|y\|_\infty^{\frac{1}{2}} \|y\|_1^{\frac{1}{2}}$.

ii) If the smallest atom is of size a, then for all $z \in L_\infty(\mu)$ we have

$$\|z\|_{2,1} \leq c_3(1 + \sqrt{\log(\mu(\Omega)/a)})\|z\|_2$$

iii) If there are N atoms, then for all $z \in L_\infty(\mu)$ we have $\|z\|_{2,1} \leq \sqrt{N}\|z\|_2$.

Proof. i) We have that

$$\begin{aligned}
\|y\|_{2,1} &= \int_0^{\|y\|_\infty} \sqrt{\mu(|y| > t)}dt \\
&\leq \left(\int_0^{\|y\|_\infty} dt\right)^{\frac{1}{2}} \left(\int_0^{\|y\|_\infty} \mu(|y| > t)dt\right)^{\frac{1}{2}} \\
&= \|y\|_\infty^{\frac{1}{2}} \|y\|_1^{\frac{1}{2}}
\end{aligned}$$

ii) We have

$$\begin{aligned}\|z\|_{2,1} &= \frac{1}{2}\int_0^\infty \frac{z^*(s)}{\sqrt{s}}ds \\ &\le \sqrt{a}\|z\|_\infty + \frac{1}{2}\int_a^{\mu(\Omega)} \frac{z^*(s)}{\sqrt{s}}ds \\ &\le \|z\|_2 + \frac{1}{2}\left(\int_a^{\mu(\Omega)} \frac{ds}{s}\right)^{\frac{1}{2}} \left(\int_a^{\mu(\Omega)} (z^*(s))^2\, ds\right)^{\frac{1}{2}} \\ &\le c_3(1+\sqrt{\log(\mu(\Omega)/a)})\|z\|_2\end{aligned}$$

iii) Let $B_1, B_2, \ldots, B_N$ be the atoms of Ω arranged so that $z^*(n)$, the value of $|z|$ on B_n, is in non-increasing order. Also, let $z^*(N+1)=0$. Then

$$\begin{aligned}\|z\|_{2,1} &= \sum_{n=1}^{N}\left(\sum_{m=1}^{n}\mu\left(B_m\right)\right)^{\frac{1}{2}}\left(z^*(n)-z^*(n+1)\right) \\ &\le \sqrt{N}\left(\sum_{n=1}^{N}\sum_{m=1}^{n}\mu\left(B_m\right)\left(z^*(n)-z^*(n+1)\right)^2\right)^{\frac{1}{2}} \\ &\le \sqrt{N}\left(\sum_{n=1}^{N}\mu\left(B_n\right)\left(z^*(n)\right)^2\right)^{\frac{1}{2}} \\ &= \sqrt{N}\|z\|_2\end{aligned}$$

as desired. □

We remark that Lemma 9(i) is a well known interpolation result, and is true for all measure spaces.

Lemma 2.7.11. *If $(\Omega, \mathcal{F}, \mu)$ is a probability space with Ω finite, then*

$$\left(\sum_{s=1}^{S}\|y_s\|_{2,1}^2\right)^{\frac{1}{2}} \le \left\|\sum_{s=1}^{S}|y_s|\right\|_\infty$$

Proof. This follows straight away from Lemma 9(i). □

Lemma 10 is also well known (and true for all probability spaces). In fact it is a reformulation of the statement that the canonical embedding $C(\Omega) \to L_{2,1}(\mu)$ has $(2,1)$ summing norm equal to 1 .

Lemma 2.7.12. *There is a constant c_4 such that, if $(\Omega, \mathcal{F}, \mu)$ is a probability space with at most N atoms, then*

$$\left(\sum_{s=1}^{S}\|z_s\|_{2,1}^2\right)^{\frac{1}{2}} \le c_4\sqrt{\log N}\left\|\left(\sum_{s=1}^{S}|z_s|^2\right)^{\frac{1}{2}}\right\|_\infty$$

Proof. Let $A \subset \Omega$ be the union of those atoms of measure less than $\frac{1}{N^2}$, so that $\mu(A) \leq \frac{1}{N}$. By Lemma 9(ii), we have that $\left\|z_s\chi_{\Omega\setminus A}\right\|_{2,1} \leq c_3\sqrt{\log N}\left\|z_s\right\|_2$, and by Lemma 9(iii), we have that $\left\|z_s\chi_A\right\|_{2,1} \leq \sqrt{N}\left\|z_s\chi_A\right\|_2$. Thus, we have that

$$\begin{aligned}\left(\sum_{s=1}^{S}\|z_s\|_{2,1}^2\right)^{\frac{1}{2}} &\leq \left(\sum_{s=1}^{S}\left\|z_s\chi_{\Omega\setminus A}\right\|_{2,1}^2\right)^{\frac{1}{2}} + \left(\sum_{s=1}^{S}\|z_s\chi_A\|_{2,1}^2\right)^{\frac{1}{2}} \\ &\leq c_3\sqrt{\log N}\left(\sum_{s=1}^{S}\|z_s\|_2^2\right)^{\frac{1}{2}} + \sqrt{N\mu(A)}\left\|\left(\sum_{s=1}^{S}|z_s|^2\right)^{\frac{1}{2}}\right\|_\infty \\ &\leq c_4\sqrt{\log N}\left\|\left(\sum_{s=1}^{S}|z_s|^2\right)^{\frac{1}{2}}\right\|_\infty\end{aligned}$$

as desired.

□

Proof of Proposition 7: Without loss of generality, we may suppose that $N = 2^{2^k}$. We prove the result by induction over k. Suppose that Ω has $2^{2^{k+1}}$ atoms. Apply Lemma 8 to cover Ω by 2^{2^k} subsets, and to give $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_S, y_1, y_2, \ldots, y_S, z_1, z_2, \ldots, z_S$ as described in the lemma. Then, by the triangle inequality

$$\left(\sum_{s=1}^{S}\|x_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq \left(\sum_{s=1}^{S}\|\bar{x}_s\|_{2,1}^2\right)^{\frac{1}{2}} + \left(\sum_{s=1}^{S}\|y_s\|_{2,1}^2\right)^{\frac{1}{2}} + \left(\sum_{s=1}^{S}\|z_s\|_{2,1}^2\right)^{\frac{1}{2}}$$

By the induction hypothesis,

$$\left(\sum_{s=1}^{S}\|\bar{x}_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq c_2 k$$

By Lemmas 10 and 11 we have that

$$\left(\sum_{s=1}^{S}\|y_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq c_1$$

and

$$\left(\sum_{s=1}^{S}\|z_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq c_1c_4 2^{-\frac{k}{2}}\left(1 + \sqrt{\log 2^{2^k}}\right) \leq 2c_1c_4$$

Hence

$$\left(\sum_{s=1}^{S} \|x_s\|_{2,1}^2\right)^{\frac{1}{2}} \leq c_2(k+1)$$

as required, taking $c_2 = 1 + 2c_1c_4$.

□

To prove the main result is now easy.

Proof of Theorem 2: By Theorem 5, it is sufficient to show that for any probability measure μ on $\{1, 2, \ldots, N\}$, the cotype 2 constant of the canonical embedding $l_\infty^N \to L_{2,1}(\mu)$ is bounded by some universal constant times $\log\log N$. But this is precisely what Proposition 7 says. □

2.7.3 Final Remarks

There is a similar result for Gaussian cotype (see[61]).

Theorem 2.7.13. *There is a constant c such that, for any operator $T : l_\infty^N \to Y$, where Y is a Banach space, the Gaussian cotype 2 constant, $\beta^{(2)}(T)$, is bounded according to the relation:*

$$\beta^{(2)}(T) \leq c\sqrt{\log\log N}\pi_{2,1}(T)$$

This result is the best possible, as is implicitly shown in [62].

Theorem 2.7.14. *There is a constant c such that for any integer N, there is an operator $T : l_\infty^N \to Y$, where Y is a Banach space, such that*

$$\beta^{(2)}(T) \geq c^{-1}\sqrt{\log\log N}\pi_{2,1}(T)$$

Since the Rademacher cotype 2 constant is greater than a constant times the Gaussian cotype 2 constant, we have the following corollary.

Corollary 2.7.15. *There is a constant c such that for any integer N, there is an operator $T : l_\infty^N \to Y$, where Y is a Banach space, such that*

$$K^{(2)}(T) \geq c^{-1}\sqrt{\log\log N}\pi_{2,1}(T)$$

We also have the following, the result originally stated in [62].

Corollary 2.7.16. *There is an operator $T : C(K) \to Y$, where Y is a Banach space, that is $(2,1)$-summing, but does not have Rademacher cotype 2.*

If we write R_N for the supremum of $K^{(2)}(T)/\pi_{2,1}(T)$ over all $T : l_\infty^N \to Y$, then we have shown that $c^{-1}\sqrt{\log\log N} \leq R_N \leq c\log\log N$. Clearly, we are left with the following problem.

Open Question. What is the asymptotic behavior of R_N ?

2.7.4 Acknowledgements

The main result of this paper originally appears in the Ph.D. thesis of the first named author [Mo1], and he would like to express his thanks to his advisor, D.J.H. Garling, and to the Science and Engineering Council who financed his studies. He would also like to express his gratitude to G.J.O. Jameson who first suggested the problem to him, and gave him much encouragement.

2.8 Remarks on Talagrand's deviation inequality for Rademacher functions

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2.8.1 Introduction.

Recently Talagrand [64] estimated the deviation of a function on $\{0,1\}^n$ from its median in terms of the Lipschitz constant of a convex extension of f to ℓ_2^n; namely, he proved that

$$P(|f - M_f| > c) \leq 4e^{-t^2/4\sigma^2}$$

where σ is the Lipschitz constant of the extension of f and P is the natural probability on $\{0,1\}^n$.

Here we extend this inequality to more general product probability spaces; in particular, we prove the same inequality for $\{0,1\}^n$ with the product measure $((1-\eta)\delta_0 + \eta\delta_1)^n$. We believe this should be useful in proofs involving random selections. As an illustration of possible applications we give a simple proof (though not with the right dependence on ε) of the Bourgain, Lindenstrauss, Milman result [65] that for $1 \leq r < s \leq 2$ and $\varepsilon > 0$, every n-dimensional subspace of L_s $(1+\varepsilon)$-embeds into ℓ_r^N with $N = c(r,s,\varepsilon)n$.

2.8.2 The main results.

For $i = 1, \dots, n$ let $(X_i, \|\cdot\|_i)$, be normed spaces, let Ω_i be a finite subset of X_i with diameter at most one and let P_i be a probability measure on Ω_i. Define

$$X = \left(\sum_{i=1}^{n} \oplus X_i\right)_2$$

and

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \subset X$$

and let

$$P = P^{(n)} = P_1 \times P_2 \times \cdots \times P_n$$

be the product probability measure on Ω. For a subset $A \subseteq \Omega$ and $t \in \Omega$ let

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‖*

$$\phi_A(t) = d(t,\ \mathrm{conv}\ A)$$

be the distance in X from t to the convex hull of the set A.

Theorem 2.8.1. $\boldsymbol{E}e^{\frac{1}{4}\phi_A^2(t)} \leq \frac{1}{P(A)}$.

Remark 11. *Talagrand's theorem is the special case of Theorem 1 when each Ω_i consists of two points and P_i gives weight $\frac{1}{2}$ to each of them. In the application below we use two point spaces for each Ω_i, but P_i does not assign the same mass to both points.*

Proof. We repeat Talagrand's induction argument [T]; the difference is only on the calculus level. For $n = 1$

$$\mathbf{E}e^{\frac{1}{4}\phi_A^2(t)} \leq P(A) + (1 - P(A))e^{\frac{1}{4}} \leq \frac{1}{P(A)},$$

as the maximal value of $r(r + (1-r)e^{\frac{1}{4}})$ for $0 \leq r \leq 1$ is 1. Assume the theorem holds for n and suppose that

$$A \subseteq \Omega_1 \times \cdots \times \Omega_n \times \Omega_{n+1}.$$

Set, for $w \in \Omega_{n+1}$,

$$A_w = \{t \in \Omega_1 \times \cdots \times \Omega_n\colon\ (t, w) \in A\},$$

where for $t = (t_1, \ldots, t_n) \in \Omega_1 \times \cdots \times \Omega_n$ and $w \in \Omega_{n+1}, (t, w)$ denotes $(t_1, \ldots, t_n, w)$. Let $v \in \Omega_{n+1}$ be such that

$$P^{(n)}(A_v) = \max_{w \in \Omega_{n+1}} P^{(n)}(A_w).$$

We shall use the following two inequalities:

$$\begin{aligned}
&\phi_A^2(t, v) \leq \phi_{A_v}^2(t) \quad \text{for all} \qquad t \in \Omega_1 \times \cdots \times \Omega_n \qquad (1)\\
&\phi_A^2(t, w) \leq \inf_{0 \leq \alpha \leq 1}[\alpha\phi_{A_w}^2(t) + (1-\alpha)\phi_{A_v}^2(t) + (1-\alpha)^2] \qquad (2)\\
&\qquad\qquad \text{for all } t \in \Omega_1 \times \cdots \times \Omega_n \text{ and for } w \neq v.
\end{aligned}$$

Using (1) and (2), Hölder's inequality and the induction hypothesis (in this order) we get

$$\begin{aligned}
&\mathbf{E}e^{\frac{1}{4}\phi_A^2(s)} = P_{n+1}\{v\}\mathbf{E}e^{\frac{1}{4}\phi_{A_v}^2(t)} + \sum_{w \neq v} P_{n+1}\{w\} \inf_{0 \leq \alpha \leq 1} \mathbf{E}\Big(e^{\frac{1}{4}\phi_{A_w}^2(t)}\Big)^{\alpha}\Big(e^{\frac{1}{4}\phi_{A_v}^2(t)}\Big)^{1-\alpha} e^{\frac{1}{4}(1-\alpha)^2}\\
&\leq P_{n+1}\{v\}\frac{1}{P^{(n)}(A_v)} + \sum_{w \neq v} P_{n+1}\{w\} \inf_{0 \leq \alpha \leq 1}\Big(\frac{1}{P^{(n)}(A_w)}\Big)^{\alpha}\Big(\frac{1}{P^{(n)}(A_v)}\Big)^{1-\alpha} e^{\frac{1}{4}(1-\alpha)^2} \qquad (3)
\end{aligned}$$

$$= \frac{1}{P^{(n)}(A_v)}\Big[P_{n+1}(v) + \sum_{w\neq v} P_{n+1}(w) \inf_{0\le\alpha\le 1}\Big(\frac{P^{(n)}(A_v)}{P^{(n)}(A_w)}\Big)^{\alpha} e^{\frac{1}{4}(1-\alpha)^2}\Big].$$

For $0 \le \lambda \le 1$, let $\alpha(\lambda)$ be the point where $\min\limits_{0\le\alpha\le 1} \frac{1}{\lambda^\alpha} e^{\frac{1}{4}(1-\alpha)^2}$ in attained; i.e.,

$$\alpha(\lambda) = \begin{cases} 1 + 2\log\lambda & \text{if } 2\log\lambda > -1 \\ e^{\frac{1}{4}} & \textit{otherwise} \end{cases}$$

and set

$$g(\lambda) = \frac{1}{\lambda^{\alpha(\lambda)}} e^{\frac{1}{4}(1-\alpha(\lambda))^2} = \begin{cases} e^{-\log\lambda - (\log\lambda)^2} & \text{if } 2\log\lambda > -1 \\ e^{\frac{1}{4}} & \textit{otherwise} \end{cases}$$

From (3) we get

$$\mathbf{E}e^{\frac{1}{4}\phi_A^2(s)} \le \frac{1}{P^{(n)}(A_v)}\Big[P_{n+1}(v) + \sum_{w\neq v} P_{n+1}(w) g(\lambda_w)\Big] \qquad (4)$$

where

$$\lambda_w = \frac{P^{(n)}(A_w)}{P^{(n)}(A_v)}.$$

□

Claim: $g(\lambda) \le 2 - \lambda$ for $0 \le \lambda \le 1$.

Using the claim we get from (4) that

$$\mathbf{E}e^{\frac{1}{4}\phi_A^2(s)} \le \frac{1}{P^{(n)}(A_v)}(q + (1-q)(2-t)) \qquad (5)$$

where $q = P_{n+1}(v)$ and $t = \frac{P(A) - P_{n+1}(v)P^n(A_v)}{(1-P_{n+1}(v))P^n(A_v)}$ (note that $0 \le q, t \le 1$). As

$$\frac{1}{P(A)} = \frac{1}{P^{(n)}(A_v)}\,\frac{1}{q + (1-q)t}, \qquad (6)$$

it suffices to prove that

$$q + (1-q)(2-t) \le \frac{1}{q + (1-q)t} \qquad (7)$$

for all $0 \le q, t \le 1$, which is easily checked. The proof of the claim is elementary: Let

$$f(\lambda) = g(\lambda) + \lambda - 2.$$

Then $f(1) = f'(1) = 0$ and $f''(\lambda) \le 0,\ 0 \le \lambda \le 1$, which implies that $f(\lambda) \le 0$ for $0 \le \lambda \le 1$.

Remark 12. *Let $2 < p < \infty$ and consider Ω as a subset of $(\sum_{i=1}^{n} \oplus X_i)_p$. Set*

$$\phi_{A,p}(t) = \inf \left\{ \Big(\sum_{i=1}^{n} \|t_i - s_i\|_i^p \Big)^{1/p} ; s = (s_1, \dots, s_n) \in \text{ conv } A \right\}.$$

Then, as pointed out by Talagrand, we also get

$$\mathbf{E} e^{\frac{1}{4}\phi_{A,p}^p(t)} \le \frac{1}{P(A)}$$

because $\phi_{A,p}^p(t) \le \phi_A^2(t)$.

Corollary 2.8.2. *Let $2 \le p < \infty$ and let f be a real convex function on $(\sum_{i=1}^{n} \oplus X_i)_p$ (it suffices to assume that f is defined on conv Ω). Let σ_p be the Lipschitz constant of f. Then, for all $c > 0$,*

$$P(|f - M_f| > c) \le 4e^{-c^p/4\sigma_p^p} \tag{8}$$

where M_f is the median of f. A similar inequality (with absolute constants replacing the two fours) holds with expectation replacing the median:

$$P(|f - \boldsymbol{E} f| > c) \le K e^{-\delta c^p/\sigma_p^p}. \tag{9}$$

(One can take $K = 8, \delta = \frac{1}{32}$).

The proof of the first assertion is identical to the proof of Theorem 3 in Talagrand's paper[64]. The second assertion follows from the first; see [66, p.142].

Remark 13. *Inequality (9) easily extends to the more general setting where each P_i is a Radon probability on B_{X_i}.*

Remark 14. *In inequalities (8) and (9) the left hand side involves only the values of f on Ω while the right hand side involves, through σ_p, the values of f on conv Ω. Thus one can replace σ_p by the infinum of the Lipschitz constants of all convex extensions of $f|_\Omega$ to conv Ω. We do not know how to compute this infimum even in the original setup of Talagrand's theorem where each Ω_i is a two point set.*

2.8.3 An application.

Lemma 2.8.3. *Let μ be a probability measure on $\{1, \dots, N\}$. Let $0 < r < s \le 2r$ and let X be an n-dimensional subspace of $L_r(\{1, \dots, N\}, \mu)$ such that $\|x\|_s \le K\|x\|_r$ for all $x \in X$ ($\|\cdot\|_s$ denotes the $L_s(\{1, \dots, N\}, \mu)$ norm). Assume moreover that $\mu(i) \le \frac{2}{N}$ for all $1 \le i \le N$. Then, for all $1 > \varepsilon > 0$ and all $k \ge c\varepsilon^{-r} r^{1/p} (\log \frac{2}{\varepsilon})^{1/p} K^r n^{1/p} N^{1/q}$, where $q = \frac{s}{r}$, $p = \frac{q}{q-1}$, there exists a subset $A \subseteq \{1, \dots, N\}$ of cardinality k such that the restriction to A is a multiple of a $(1+\varepsilon)$-isomorphism on X. In particular, X $(1+\varepsilon)$-embeds into ℓ_r^k.*

Proof. Let $\delta_i, i = 1, \dots, N$, be independent mean δ 0,1-valued random variables. Fix $x \in X, \|x\|_r = 1$ and define $f : \mathbf{R}^N \to \mathbf{R}$ by

$$f((a_i)_{i=1}^N) = \sum_{i=1}^N a_i \mu(i)|x(i)|^r.$$

Then

$$\begin{aligned}\sigma_p(f) &= \sup_{\sum_{i=1}^N |a_i|^p = 1} \sum_{i=1}^N a_i \mu(i)|x(i)|^r \\ &= \Big(\sum_{i=1}^N \mu(i)^q |x(i)|^s\Big)^{1/q} \le \Big(\frac{2}{N}\Big)^{\frac{q-1}{q}} \Big(\sum_{i=1}^N \mu(i)|x(i)|^s\Big)^{\frac{r}{s}} \\ &\le \Big(\frac{2}{N}\Big)^{1/p} K^r.\end{aligned}$$

It follows from Corollary 4 that

$$P\Big(\Big|\sum_{i=1}^N \delta_i \mu(i)|x(i)|^r - \delta\Big| > c\Big) \le 4e^{-c^p N/8K^{rp}} \tag{10}$$

and, consequently, using the usual estimate on the size of an ε-net in ∂B_X; cf. [MS] p. 7, that

$$P\Big(\Big|\sum_{i=1}^N \delta_i \mu(i)|x(i)|^r - \delta\Big| \le \varepsilon^r \delta \text{ for all } x \in \partial B_X\Big) \ge 1 - 4\exp\Big(nr\log\frac{2}{\varepsilon} - \varepsilon^{rp}\delta^p N/8K^{rp}\Big).$$

Set $k = 2\delta N$ (= twice the average size of $\{i; \delta_i = 1\}$). Then, for $\eta = c\varepsilon^{rp}\Big(r\log\frac{2}{\varepsilon}\Big)^{-1}$ (c universal) and $n \le \eta\delta^p N/K^{rp}$, the probability above is larger then $\frac{1}{2}$, so we can find a set of cardinality k which satisfies the requirement. Eliminating δ from the two equations $k = 2\delta N$ and $n = \eta\delta^p N/K^{rp}$ we get

$$k \approx 2\eta^{-1/p} n^{1/p} N^{1/q} K^r.$$

□

Theorem 2.8.4. *[65]. Let* $0 < r < t \le 2$ *and let* $T, \varepsilon > 0$*. Then there exists a constant* $C = C(\varepsilon, T, r, t)$ *such that any* n *dimensional subspace* X *of* L_r *with type* t *constant* K, $(1+\varepsilon)$*-embeds into* ℓ_r^N *with* $N \le C \cdot n$.

Proof. We may assume that $X \subseteq L_r^M$ for some finite M. By the Maurey-Nikishin-Rosenthal factorization theorem ([M] Theorem 8 and Proposition 44), there exists a probability measure μ on $\{1, \dots, M\}$ such that $\|x\|_s \le K\|x\|_r$ for all $x \in X$ where $s = \frac{r+t}{2}$ and K depends on r, t and T only. Splitting the large atoms of μ into ones with measure $\le \frac{1}{M}$ we get a new probability measure $\bar{\mu}$ on $\{1, \dots, N\}$ with $N \le 2M$ and $\bar{\mu}(i) \le \frac{1}{M} \le \frac{2}{N}$. $L_u(\mu)$ embeds isometrically (as a sublattice) into $L_u(\bar{\mu})$ for all u and the inequality $\|x\|_s \le K\|x\|_r$ for all $x \in X$ (in the new embedding) stays true. Applying Lemma 7 we get that X $(1+\delta)$-embeds into L_r^k where k is of order $n^{1/p}M^{1/q}$.

Using the Maurey-Nikishin-Rosenthal theorem again we may repeat the argument to get that X $(1+\delta)^2$-embeds into L_r^k for k of order $n^{\frac{1}{p}+\frac{1}{pq}}M^{\frac{1}{q^2}}$. Iterating one gets the result. This part is the same as in [65]. (One should be more careful than we have been above, taking the exact form of k into account, but it works.) $\square$

Remark 15. *Both B. Maurey and M. Talagrand pointed out to us that versions of inequality (10) follow from known inequalities; in particular, (10) is an immediate consequence of the Azuma-Pisier inequality (see p. 45 in[66]) except that the exponent on the right side of (10) must be multiplied by a constant δ_p which tends to infinity with p.*

Since the degeneracy of this constant is unimportant for proving Theorem 8, we in fact do not have a good application of (our slight generalization of) Talagrand's isoperimetric inequality.

On the other hand, it is possible that the approach outlined above can be used for general subspaces of L_r, in which case one expects to use a version of Lemma 7 with "s" close to "r", which forces "p" to infinity.

2.9 Integral Operators on Spaces of Continuous Vector-valued functions

Paulette Saab

Abstract:
Let X be a compact Hausdorff space, let E be a Banach space, and let $C(X, E)$ stand for the Banach space of E-valued continuous functions on X under the uniform norm. In this paper we characterize Integral operators (in the sense of Grothendieck) on $C(X, E)$ spaces in term of their representing vector measures. This is then used to give some applications to Nuclear operators on $C(X, E)$ spaces.

AMS(MOS) subject Classification (1980). Primary 46E40, 46G10; Secondary 28B05, 28B20.

2.9.1 Introduction.

Let X be a compact Hausdorff space, let E and F be Banach spaces. Denote by $C(X, E)$ the space of all continuous E-valued functions defined on X under the uniform norm. In[67] C. Swartz showed that a bounded linear operator $T : C(X, E) \rightrightarrows F$ with representing measure G is absolutely summing if and only if each of the values of G is an absolutely summing operator from E to F and G is of bounded variation as a measure taking its values in the space of absolutely summing operators from E to F equipped with the absolutely summing operators norm. In this paper we shall extend Swartz's result to the class of (Grothendieck) absolutely summing operator on $C(X, E)$ spaces. More precisely we shall show that a bounded linear operator $T : C(X, E) \to F$ with representing measure G is an integral operator if and only if each of the values of G is an bounded variation from E to F and G is of bounded variation as a vector measure taking its values in the space of absolutely summing operator from E to F equipped with the integral norm. This result is then used to give some applications to Nuclear operators on $C(X, E)$ spaces.

2.9.2 Preliminaries.

If X is a compact Hausdorff space and E is a Banach space, then $C(X, E)$ will denote the Banach space of all continuous E-valued functions equipped with the uniform norm. It is well known [4,page 182] that the dual of $C(X, E)$ is isometrically isomorphic to the space $M(X, E^*)$ of all regular E^*-valued measures on X that are of bounded variation. When E is the scalar field, we will simply write $C(X)$ and $M(X)$ for $C(X, E)$ and $M(X, E^*)$. If $\mu \in M(X, E^*)$ and $e \in E$, we will denote by $|\mu|$ the variation of μ and by $\langle e, mu\rangle$ the element of $M(X)$ defined on each Borel subset B of X by

$$\langle e, \mu\rangle(B) = \mu(B)(e).$$

The duality between $M(X, E^*)$ and $C(X, E)$ is then defined as follows: for each $f \in C(X, E)$ and $e \in E$

$$\mu(f \otimes e) = \int_X f d\langle e, \mu\rangle$$

where $f \otimes e$ is the element of $C(X,E)$ defined by

$$f \otimes e(x) = f(x)e \text{ for all } x \in X.$$

If B is a Borel subset of X, then 1_B will denote the characteristic function of B, and if $e \in E$ we let 1_B denote the element of $C(X,E^{**})$ defined by

$$1_B(\mu) = \langle e, \mu\rangle(B) = \mu(B)(e)$$

for each $\mu \in M(X,E^*)$.

If X is a a compact Hausdorff space , E and F are Banach space, every bounded linear operator $T : C(X,E) \rightrightarrows F$ has a representing measure G. The measure G is defined on the σ-field Σ of Borel subsets of X and takes its values in$\mathcal{L}(E,F^{**})$, the space of all bounded linear operator from E to F^{**}. The measure G is such that for each Borel subset B of X and for each $e \in E$

$$G(B)e = T^{**}(1_B)$$

For $y^* \in F^*$, if we denote by G_{y^*} the E^*- valued measure on X such that for each Borel subset B of X and each $e \in E$

$$\langle e, G_{y^*}\rangle(B) = \langle y^*, G(B)e\rangle$$

then G_{y^*} is the unique element of $M(X,E^*)$ that represents T^*y^* in the sense that for each $f \in C(X,E)$

$$\langle y^*, Tf\rangle = \int_X f(x)dG_{y^*}(x)$$

If E ad F are Banach space, we denote by $E \otimes_\epsilon F$ the algebraic tensor product of E and F endowed with the norm $||.||_\epsilon$

$$\left\| \sum_{i=1}^{m} x_i \otimes y_i \right\|_\epsilon = \sup\left\{ \left| \sum_{i=1}^{m} x^*(x_i)y^*(y_i) \right| \quad | \quad \|x^*\|, \|y^*\| \leq 1 \right\}.$$

The completion $E\hat{\otimes}_\epsilon F$ of $E \otimes_\epsilon F$ is called the injective tensor product of E and F.

A bounded linear operator $T :\ E \to F$ from a Banach space E into a Banach space F is said to be an **integral operator** if the bilinear form τ on $E \times F^*$ defined by $\tau(e,y^*) = y^*(Te)$ for $e \in E$ and $y^* \in F^*$ determines an element of $(E\hat{\otimes}_\epsilon F^*)^*$, the dual of the injective tensor product $E\hat{\otimes}_\epsilon F^*$ of the Banach spaces E and F^*. The integral norm of T, which we will denote by $\| T \|_{\text{int}}$, is just the norm of the bounded linear functional induced by τ as an element of the dual space $(E\hat{\otimes}_\epsilon F^*)^*$. Hence if $T :\ E \longrightarrow F$ is an integral operator, then

$$\| T \|_{\text{int}} = \sup\left\{ |\sum_{i=1}^{n} y_i^*(Te_i)| : \| \sum_{i=1}^{n} e_i \otimes y_i^* \|_\epsilon \leq 1 \right\}.$$

Finally, $I(E,F)$ will stand for the bounded variation of all integral operators from E to F equipped with the integral norm. For all undefined notions and notations we refer the reader to [68–70]

One of the most useful tools in the study of an integral operators T between two Banach spaces E and F is its representation by a finite regular Borel scalar measure on the compact $B(E^*) \times B(F^{**})$, the product of the closed unit balls of E^* and F^{**} equipped with their weak*-topologies. This, of course, goes back to Grothendieck [6] and can be used to characterize integral operators by the following characterization that we will state and prove before proving the main result.

Proposition 2.9.1. *A bounded linear operator* $T: E \to F$ *between two Banach spaces is an integral operator if and only if there exists a regular* F^{**}*-valued vector measure* m *of bounded variation defined on the* σ*-field of Borel subsets of the closed unit ball* $B(E^*)$*, such that for each* $e \in E$

$$Te = \int_{B(E^*)} e^*(e)\ dm(e^*)$$

In this case m *can be chosen so that* $\|T\|_{int} = |m|(B(E^*))$.

Proof. Assume that there exists a regular F^{**}-valued vector measure m defined on the σ-field of Borel subsets of $B(E^*)$ with $|m(B(E^*)) < \infty$ and such that for each $e \in E$

$$Te = \int_{B(E^*)} e^*(e)\ dm(e^*)$$

This in particular shows that the operator $J \circ T : E \to F^{**}$, where J denotes the natural embedding of F into F^{**} , has an extension $\hat{T}$ to an integral operator from $C(B(E^*))$ to F^{**}. Hence T is an integral operator [69, p.233] and

$$\|T\|_{int} \leq \|T\|_{int}\hat{T} = |m(B(E^*)).$$

Conversely, suppose that $T : E \to F$ is an integral operator, then it follows from [69, p.231] that there exists a regular Borel measure μ on $B(E^*) \times B(F^{**})$ such that for each $e \in E$ and $y^* \in F^*$

$$\langle y^*, Te \rangle = \int_{B(E^*)\times B(F^{**})} e^*(e)y^{**}(y^*)d\mu(e^*, y^{**})$$

and

$$\|T\|_{int} = |\mu|(B(E^*) \times B(F^{**})).$$

Following [69, p.234], define

$$S : E \to L_\infty \quad \text{by} \quad Se(e^*, y^{**}) = e^*(e)$$

and

$$R : F^* \to L_\infty \quad \text{by} \quad Ry^*(e^*, y^{**}) = y^{**}(y^*).$$

Then S and R are bounded linear operators with $\|S, \|\|R\| \leq 1$.
Let Q be the restriction of R^* to L_{1_μ}, then Q is a bounded linear operators from L_{1_μ} into F^{**}. It is immediate that $J \circ T = Q \circ I \circ S$ where $I : L_{\infty_\mu} \to L_{1_\mu}$ is the natural inclusion and $J : F \to F^{**}$ is the natural embedding.

This in particular shows that the operator $Q \circ I$ restricted to $C(B(E^*) \times B(F^{**}))$ is an integral operator whose representing F^{**}-valued measure $\hat{\mu}$ is such that

$$\hat{\mu}(C) = Q(1_C)$$

for each Borel subset C of $B(E^*) \times B(F^{**})$. Finally note that since the natural projection of $B(E^*) \times B(F^{**})$ onto $B(E^*)$ is a continuous mapping it induces a bounded linear operators from $C(B(E^*))$ into $C(B(E^*) \times B(F^{**}))$ as follows; for $\phi \in C(B(E^*))$, let $\hat{\phi} \in C(B(E^*) \times B(F^{**}))$ be such that

$$\hat{\phi}(e^*, y^{**}) = \phi(e^*).$$

This of course shows that the operator $J \circ T$ extends to an integral operator

$$\hat{T} : C(B(E^*)) \to F^{**}$$

such that for each $\phi \in C(B(E^*))$

$$\hat{T}\phi = Q \circ I(\hat{\phi})$$

It is immediate that the F^{**}-valued measure m representing $\hat{T}$ is such that

$$m(B) = Q(1_{B \times B(F^{**})})$$

and

$$|m|(B(E^*) \leq |\mu|(B(E^*) \times B(F^{**})) = \|T\|_{int}.$$

Since $\hat{T}$ extends T it follows that

$$\|T\|_{int} = |m|(B(E^*)).$$

□

2.9.3 Main Result.

Throughout this section X is a compact Hausdorff space, E and F are Banach space and $T : C(X, E) \to F$ is a bounded linear operators with representing measure G. The main result of this paper gives a characterization of integral operator T in terms of some properties of G. The first step to achieve such a characterization is to show that if T is an integral operator on $C(X, E)$, then one can do a little better than Proposition 1 by representing the operator T by a regular F^{**}-valued measure of bounded variation defined on the σ-field of Borel subsets of $X \times B(E^*)$ rather than on the whole unit ball of $(C(X, E))^*$. The proof we present here is different from our earlier proof which relied on our result [71]. We would like to thank the referee for suggesting the following approach which relies more on basic knowledge and classical results of vector measures that can be found in [69, 70].

Theorem 2.9.2. *If $C(X, E) \to F$ is an integral operator, then there exists a regular F^{**}-valued measure of f bounded variation θ on the σ-field of Borel subsets of $X \times B(E^*)$ such that for each $f \in C(X, E)$ and $y^* \in F^*$*

$$\langle y^*, Tf \rangle = \int_{X \times B(E^*)} e^*(f(x)) d\theta(x, e^*)$$

and

$$\|T\|_{int} = |\theta|(X \times B(E^*)).$$

Proof. Suppose that $T : C(X, E) \longrightarrow F$ is an integral operator, then the bilinear map $\tau(f, y^*) = y^*(T(f))$ for $f \in C(X, E)$ and $y^* \in F^*$ defines an element of $(C(X, E) \otimes_\epsilon F^*)^*$. It is easy to check that $C(X, E) \otimes_\epsilon F^*$ embeds isometrically in $C(X \times B(E^*) \times B(F^{**}))$ such that if $f \in C(X, E)$ and $y^* \in F^*$

$$f \otimes y^* (x, e^*, y^{**}) = e^*(f(x)) y^{**}(y^*)$$

for each $(x, e^*, y^{**}) \in \Omega = X \times B(E^*) \times B(F^{**}$. Hence by the Hahn-Banach theorem there exists a regular Borel measure μ on Ω so that for each $f \in C(X, E)$ and $y^* \in F^*$

$$\langle y^*, Tf \rangle = \int_\Omega e^*(f(x)) y^{**}(y^*) d\mu(x, e^*, y^{**}).$$

and

$$\|T\|_{\text{int}} = |\mu|(\Omega)$$

The proof now follows the steps of Proposition 1. Indeed, since the natural projection of Ω onto $X \times B\left(E^*\right)$ is continuous it induces a bounded linear operator

$$S : C\left(X \times B\left(E^*\right)\right) \longrightarrow L_\infty(\mu)$$

such that if $\phi \in C\left(X \times B\left(E^*\right)\right)$ then

$$S(\phi)\left(x, e^*, y^{**}\right) = \phi\left(x, e^*\right)$$

for all $\left(x, e^*, y^{**}\right) \in \Omega$. Let $R : F^* \longrightarrow L_\infty(\mu)$ be defined by $R\left(y^*\right)\left(x, e^*, y^{**}\right) = y^{**}\left(y^*\right)$ and let Q be the restriction of R^* to $L_1(\mu)$. It is straightforward to check that the operator $Q \circ I \circ S$ is an integral operator whose representing measure θ is such that

$$\theta(C) = Q\left(1_{C \times B(F^{**})}\right)$$

for each Borel subset C of $X \times B\left(E^*\right)$, and

$$|\theta|\left(X \times B\left(E^*\right)\right) \leq |\mu|(\Omega) = \|T\|_{\text{int}}\,.$$

Moreover note that $C(X,E)$ embeds isometrically in $C\left(X \times B\left(E^*\right)\right)$ by

$$f\left(x, e^*\right) = e^*(f(x))$$

for each $f \in C(X,E)$ and $\left(x, e^*\right) \in X \times B\left(E^*\right)$. It is easy to see that for each $f \in C(X,E)$

$$J \circ T(f) = Q \circ I \circ S(f)$$

hence

$$T(f) = \int_{X \times B(E^*)} e^*(f(x))d\theta\left(x, e^*\right)$$

and

$$\|T\|_{\text{int}} \leq \|Q \circ \circ S\|_{\text{int}} \leq |\theta|\left(X \times B\left(E^*\right)\right).$$

□

This complete the proof of Lemma 2 . We are now ready to state and prove the main result of this paper.

Theorem 2.9.3. *Let X be a compact Hausdorff space, let E and F be Banach spaces and let $T : C(X,E) \to F$ be a bounded linear operator with representing measure G. Then T is an integral operator if and only if for each Borel subset B of X, the operator $G(B) : E \longrightarrow F$ is integral and the measure G is of finite variation as a vector measure taking its values in $I(E,F)$ equipped with the integral norm.*

Proof. Assume that $T : C(X,E) \to F$ is an integral operator. Let G denote the vector measure representing the operator T, hence for each Borel subset B of X and each $e \in E$

$$G(B)e = T^{**}\left(1_B \otimes e\right)$$

It is clear at this stage, that since T^{**} is also integral [69, p.236], then for each Borel subset B of X the operator $G(B) : E \longrightarrow F$ is an integral operator as the composition of T^{**} and the bounded linear operator $E \longrightarrow C(X,E)^{**}$ which to each e in E associates the element $1_B \otimes e$. In what follows we shall concentrate on estimating the value of $\|G(B)\|_{\text{int}}$. For this note that by Lemma 2 there exists a regular F^{**}-valued measure θ defined on the σ-field of Borel subsets of $X \times B\left(E^*\right)$ such that, for each $f \in C(X,E)$

$$T(f) = \int_{X \times B(E^*)} e^*(f(x)) d\theta\left(x, e^*\right)$$

and

$$\|T\|_{\text{int}} = |\theta|\left(X \times B\left(E^*\right)\right).$$

It follows that for each $y^* \in F^*$ and $f \in C(X,E)$

$$\left\langle T^* y^*, f\right\rangle = \int_{X \times B(E^*)} e^*(f(x)) d\theta_{y^*}\left(x, e^*\right).$$

where θ_{y^*} is the scalar measure on $X \times B\left(E^*\right)$ defined by

$$\theta_{y^*}(C) = \theta(C)\left(y^*\right)$$

for each Borel subset C of $X \times B\left(E^*\right)$. We claim that for each Borel subset B of X and for each $e \in E$

$$G(B)e = \int_{B \times B(E^*)} e^*(e) d\theta\left(x, e^*\right).$$

For this suppose that $y^* \in F^*$ and K is a compact G_δ subset of X. Let $\left(u_n\right)_{n \geq 1}$ be a sequence of continuous real valued functions so that $0 \leq u_n \leq 1$ for all $n \geq 1$ and u_n converges to 1_K pointwise. If $e \in E$, then

$$\begin{aligned}
\left\langle G(K)e, y^*\right\rangle &= \left\langle e, G_{y^*}\right\rangle(K) \\
&= \lim_{n \to \infty}\left\langle y^*, T\left(u_n \otimes e\right)\right\rangle \\
&= \lim_{n \to \infty} \int_{X \times B(E^*)} u_n(x) e^*(e) d\theta_{y^*}\left(x, e^*\right). \\
&= \int_{X \times B(E^*)} 1_K(x) e^*(e) d\theta_{y^*}\left(x, e^*\right) \\
&= \int_{K \times B(E^*)} e^*(e) d\theta_{y^*}\left(x, e^*\right)
\end{aligned}$$

Therefore

$$G(K)e = \int_{K \times B(E^*)} e^*(e) d\theta\left(x, e^*\right).$$

Moreover, since for each $y^* \in F^*$ the E^*-valued vector measure G_{y^*} is regular, it follows that

$$G(B)e = \int_{B \times B(E^*)} e^*(e) d\theta\left(x, e^*\right)$$

for all Borel subsets B of X. This proves our claim. If we denote by $p : X \times B\left(E^*\right) \longrightarrow E^*$ the projection mapping which to each $\left(x, e^*\right)$ in $X \times B\left(E^*\right)$ associates e^* in $B\left(E^*\right)$, then for each Borel subset B of X, let λ_B denote the regular F^{**}-valued measure defined on the σ-field of Borel subsets of $B\left(E^*\right)$ as follows: for each Borel subset V of $B\left(E^*\right)$

$$\lambda_B(V) = \theta(B \times V).$$

In other words $\lambda_B = \theta \mid B \times B\left(E^*\right) \circ p^{-1}$ which is the image measure of the restriction of θ to $B \times B\left(E^*\right)$ by p. This implies that for each $e \in E$ $(**)$

$$\int_{B(E^*)} e^*(e) d\lambda_B\left(e^*\right) = \int_{B\times B(E^*)} e^*(e) d\theta\left(x, e^*\right).$$

Equations (*) and (**) show that the measure λ_B is a regular F^{**}-valued measure that represents the operator $G(B) : E \longrightarrow F$. Hence by Proposition 1

$$\|G(B)\|_{\text{int}} \leq \left|\lambda_B\right|\left(B\left(E^*\right)\right) = |\theta|\left(B \times B\left(E^*\right)\right).$$

and therefore

$$(\dagger) \quad \|G(B)\|_{\text{int}} \leq |\theta|\left(B \times B\left(E^*\right)\right.$$

Hence the vector measure G representing the operator T takes its values in $I(E, F)$ and it follows from (†) that

$$|G|_{\text{int}}\,(X) \leq |\theta|\left(X \times B\left(E^*\right)\right)$$

Here of course $|G|_{\text{int}}\,(X) = \sup \sum_{B_i \in \pi} \left\|G\left(B_i\right)\right\|_{\text{int}}$ where the sup is taken over all the finite partitions of X into Borel subsets of X. Conversely, suppose $T : C(X, E) \longrightarrow F$ is such that

$$G : \Sigma \longrightarrow I(E, F) \quad \text{ and } |G|_{\text{int}}\,(X) < \infty,$$

we need to show that T is an integral operator. For this, note that if J denotes the natural embedding of F into F^{**}, then by [4, p. 233] it is enough to show that $J \circ T : C(X, E) \longrightarrow F^{**}$ is an integral operator. The mapping J induces a mapping $\hat{J} : I(E, F) \longrightarrow I\left(E, F^{**}\right)$ defined as follows, for each $U \in I(E, F)$

$$\hat{J}(U) = J \circ U$$

this in turn induces a vector measure $\hat{G} : \Sigma \longrightarrow I\left(E, F^{**}\right)$ such that

$$\hat{G}(B) = \hat{J}(G(B))$$

for all $B \in \sum$. It is immediate that $\hat{G}$ is the measure representing the operator $J \circ T$. Since $|G|_{\text{int}}\,(X) < \infty$, it follows easily that $|\hat{G}|_{\text{int}}\,(X) < \infty$. Moreover, since $I\left(E, F^{**}\right)$ is isometric to the dual space of the injective tensor product $E \hat{\otimes}_\epsilon F^* [4, p.237]$, it follows that $\hat{G}$ is a vector measure of bounded variation taking its values in the dual space $\left(E \hat{\otimes}_\epsilon F^*\right)^*$. Since $|\hat{G}|_{\text{int}}\,(X) < \infty$, it follows from [4,page 7] that $\hat{G}$ is strongly additive. Recall that for each $e \in E$ and $y^* \in F^*$, the scalar measure $\langle e, G_{y^*}\rangle$ is in $M(X)$. Hence for $u = \sum_{i=1}^n e_i \otimes y_i^* \in E \hat{\otimes}_\epsilon F^*$, let $\langle u, \hat{G}\rangle$ be the scalar measure defined on X as follows: for each Borel subset B of X

$$\begin{aligned} \langle u, \hat{G}\rangle(B) &= \hat{G}(B)(u) \\ &= \sum_{i=1}^n \langle G(B) e_i, y_i^*\rangle \\ &= \sum_{i=1}^n \left\langle e_i, G_{y_i^*}\right\rangle (B) \end{aligned}$$

This implies that for each $u \in E \hat{\otimes}_\epsilon F^*$ the measure $\langle u, \hat{G}\rangle \in M(X)$. This in particular shows that $\hat{G} : \Sigma \longrightarrow \left(E \hat{\otimes}_\epsilon F^*\right)^*$ is countably additive with respect to the weak * topology on $\left(E \hat{\otimes}_\epsilon F^*\right)^*$. Since $\hat{G}$ is strongly additive, it follows easily that G is countably additive. Finally, since $\hat{G}$ is weak k^* regular, a glance at [69, p.157] reveals that it is also regular.

It follows that $\hat{G}$ defines an element of $C\left(X, E\hat{\otimes}_\epsilon F^*\right)^*$, hence by [5, page 269] there exists a weak $^*|\hat{G}|_{\text{int}}$ – integrable function

$$h: \quad X \longrightarrow I\left(E, F^{**}\right) \simeq \left(E\hat{\otimes}_\epsilon F^*\right)^*$$

such that $\|h(x)\|_{\text{int}} = 1 \quad |\hat{G}|_{\text{int}}$ a.e., $\hat{G} = h|\hat{G}|_{\text{int}}$, and for each $f \in C\left(X, E\hat{\otimes}_\epsilon F^*\right)$

$$\langle f, \hat{G}\rangle = \int_X \langle f(x), h(x)\rangle d|\hat{G}|_{\text{int}}\,(x)$$

In particular, if $\varphi \in C(X, E)$ and $y^* \in F^*$, let $f = \varphi \otimes y^*$ be the element of $C\left(X, E\hat{\otimes}_\epsilon F^*\right)$ defined by

$$\varphi \otimes y^*(x) = \varphi(x) \otimes y^* \text{ for all } x \in X.$$

then

$$\left\langle \varphi \otimes y^*, \hat{G}\right\rangle = \int_X y^*\langle h(x), \varphi(x)\rangle d|\hat{G}|_{\text{int}} \mid (x)$$

But we also have $(***)$

$$\langle J \circ T\varphi, y^*\rangle = \left\langle \varphi, \hat{G}_{y^*}\right\rangle$$

where $\hat{G}_{y^*}$ is the element of $M\left(X, E^*\right)$ such that for each Borel subset B of X

$$\hat{G}_{y^*}(B) = y^*(\hat{G}(B)) = y^*(G(B)).$$

Hence since $\hat{G} = h \cdot |\hat{G}|_{\text{int}}$, it follows that for each Borel subset B of X

$$\hat{G}_{y^*}(B) = \int_B y^* h(x) d|\hat{G}|_{\text{int}}\,(x).$$

This implies that for each $\varphi \in C(X, E)$ $(****)$

$$\left\langle \varphi, \hat{G}_{y^*}\right\rangle = \int_X y^*\langle h(x), \varphi(x)\rangle d|\hat{G}|_{\text{int}}\,(x).$$

It follows that if $\sum_{i=1}^n \varphi_i \otimes y_i^*$ is in $C(X, E) \otimes_\epsilon F^*$ with $\|\sum_{i=1}^n \varphi_i \otimes y_i^*\|_\epsilon \leq 1$, then for each x in X

$$\left|\left\langle h(x), \sum_{i=1}^n \varphi_i(x) \otimes y_i^*\right\rangle\right| \leq \|h(x)\|_{\text{int}}$$

since

$$\left\|\sum_{i=1}^n \varphi_i(x) \otimes y_i^*\right\|_\epsilon \leq \left\|\sum_{i=1}^n \varphi_i \otimes y_i^*\right\|_\epsilon \leq 1$$

It follows then from (***) and (****) that

$$\left\|\sum_{i=1}^n \langle T\varphi_i, y_i^*\rangle\right\| \leq |\hat{G}|_{\text{int}}\,(X) < \infty$$

Hence $J \circ T$ is an element of $I\left(C(X, E), F^{**}\right)$ which implies that $T \in I(C(X, E), F)$. This completes the proof.

□

2.9.4 Applications.

The study of integral operators on $C(X,E)$ spaces was motivated by some problems that arose in [72] concerning nuclear operators on $C(X,E)$ spaces. Recall that an operator T between two Banach spaces Y and Z is said to be a nuclear operator if there exist sequences (y_n^*) and (z_n) in Y^* and Z respectively, such that $\sum_n \|y_n^*\|\,\|z_n\| < \infty$ and for each $y \in Y$

$$T_y = \sum_n y_n^*(y) z_n$$

the nuclear norm is defined by

$$\|T\|_{nuc} = \inf \left\{ \sum_n \|y_n^*\|\,\|z_n\| \right\}$$

where the infimum is taken over all sequences (y_n^*) and (z_n) such that $Ty = \sum_n y_n^*(y) z_n$ for all $y \in Y$. We shall denote by $N(Y,Z)$ the space of nuclear operators from Y to Z under the nuclear norm. The study of nuclear operators on $C(X,E)$ spaces was initiated in [73] where some of the known results in the scalar case were extended. Nuclear operators on $C(X,E)$ spaces were also considered in [74] where it was shown that if $T : C(X,E) \longrightarrow F$ is a nuclear operator then for each Borel subset B of X, the operator $G(B) : \quad E \longrightarrow F$ is nuclear. In [72] it was shown that the representing measure G of a nuclear operator is in fact of bounded variation as a vector measure taking its values in $N(E,F)$. Hence all the above known results can be summarized as follows:

Proposition 2.9.4. *Let X be a compact Hausdorff space and let E and F be two Banach spaces. If T is a nuclear operator from $C(X,E)$ into F with representing measure G, then (i) For any Borel subset B of X the operator $G(B) : E \to F$ is nuclear, and (ii) G is of bounded variation as a vector measure taking its values in $N(E,F)$ under the nuclear norm.*

Easy examples show that conditions (i) and (ii) above do not characterize nuclear operators on $C(X,E)$ spaces. As a matter of fact, it was shown in [72] that counterexamples can be given as soon as the range space F fails to have the so-called Radon-Nikodym property (RNP) see [69]. The interesting question that arises is then:

> Assume that F has the (RNP) and that $T : C(X,E) \longrightarrow F$ is an operator whose representing measure G satisfies conditions (i) and (ii) of Proposition 4, is T a nuclear operator?

In what follows, we shall show how Theorem 3 can be used to give a positive answer to Question 5 in case F is assumed to be complemented in its bidual.

Theorem 2.9.5. *Let F be a Banach space with the Radon-Nikodym property and assume that F is complemented in its bidual. A bounded linear operator $T : C(X,E) \longrightarrow F$ is nuclear whenever its representing vector measure G satisfies the following conditions (i) For each Borel subset B of X, the operator $G(B) : E \longrightarrow F$ is nuclear, and (ii) G is of bounded variation as a vector measure taking its values in $N(E,F)$.*

Proof. First note that for an arbitrary Banach space F any element U in $N(E,F)$ is in $I(E,F)$ with $\|U\|_{\text{int}} \le \|U\|_{\text{nuc}}$. Hence by Theorem 3 an operator $T : C(X,E) \longrightarrow F$ that satisfies conditions (i) and (ii) above is indeed integral. Therefore when in addition we assume that F is complemented in its bidual F^{**}, the operator T becomes Pietsch integral (also called strictly integral) see [69, p.235]. Under these conditions and if F is assumed to have the Radon-Nikodym property, it is well known [69, p.175] that T becomes a nuclear operator. This completes the proof. $\square$

Remark As we have just seen, Theorem 6 is a direct application of the main result of this paper, it should be noted that Theorem 6 gives a positive answer to Question 5 when F is a dual space and when F is a Banach lattice.

2.10 Operators which factor through Banach lattices not containing c_0

In this supplement to [81, 82], we give an intrinsic characterization of (bounded,linear) operators on Banach lattices which factor through Banach lattices not containing a copy of c_0 which complements the characterization of [81, 82] that an operator admits such a factorization if and only if it can be written as the product of two operators neither of which preserves a copy of c_0. The intrinsic characterization is that the restriction of the second adjoint of the operator to the ideal generated by the lattice in its bidual does not preserve a copy of c_0. This property of an operator was introduced by C. Niculescu [83]. under the name strong type B. That this condition is stronger than the property that the operator itself does not preserve a copy of c_0 is exhibited by the example constructed in [84] and studied further in [85]. When we wrote [81, 82] we did not know that Niculescu [83, 86, 87] was independently considering some of the same problems; we now want to acknowledge his priority for discovering some results that overlap with ours. Also, in remark 1 below, we indicate how our results, when translated into Niculescus language, answer some questions he raised.

Before stating the theorem of this note, we introduce some notation: Given a Banach lattice E, $I(E)$ (respectively, $Bd(E)$) denotes the closed order ideal (respectively, band) in E^{**} generated by E. The symbols E, F, G will be reserved for Banach lattices while X, Y, Z will be used for general Banach spaces. Terminology is as in [GJ3], and the reader will probably need a copy of [GJ3] on hand to follow the proof of the theorem.

Theorem 2.10.1. *Let E be a Banach lattice and let T be an operator from E into a Banach space X. The following are equivalent:*

(1) $T^{**}{}_{|I(E)}$ does not preserve a copy of c_0.

(2) $T^{**}{}_{|I(E)}$ does not preserve a positive disjoint copy of c_0.

(3) $T^{**}[Bd(E)] \subset X$.

(4) There exists a Banach space Y and bounded linear operators $u\colon E \to Y$ and $v\colon Y \to X$ such that neither u nor v preserves a copy of c_0 and $T = vu$.

(5) There exists a weakly sequentially complete Banach lattice F, an interval preserving lattice homomorphism $R\colon I(E) \to F$, and an operator $S\colon F \to X$ so that $T^{**}{}_{|I(E)} = SR$.

Proof. That (1) and (2) are equivalent follows from Theorem I.3 of [82] (that the other conditions in Theorem I.3 are equivalent to (2) was proved earlier by Niculescu—see Theorem 3.4 in [83]). Niculescu [[83], Proposition 4.2] stated the equivalence of (2) and (3); we follow his lead in leaving the proof as an exercise for the reader, but note that (2) is the statement we tie to the other conditions. The equivalence of (4) with (5) was proved in Corollary I.8 in [82] and (5) ⇒ (2) is immediate, so we only need to check that (2) implies (5). In fact, the proof of this implication is essentially contained in the proof of Theorem I.7 of [82], but since only weaker implications are explicitly stated there we present here a detailed proof.

Let W be the absolute convex solid hull of $T^*Ball(X^*)$ in E^* and let G be the interpolation space (cf. [82]) corresponding to W in E^*. Let $J\colon G \to E^*$ be the canonical injection and $I\colon X^* \to G$ such that $T^* = JI$. By Lemma I.5 of [82] we have for each μ in E^{**} and $n = 1, 2, \ldots,$

$$\|J^*\mu\| \le 2^n sup\,\{\, \|T^{**}\nu\| \colon\, \nu \in E^{**}, |\nu| \le |\mu| \,\} + 2^{-n}\|\mu\|. \qquad (\dagger)$$

Letting F be the norm closure of $J^*[I(E)]$ in G^* and setting $R = J^*{}_{|I(E)}$, $S = I^*{}_{|F}$, we have that $T^{**}{}_{|I(E)}$ factors through R and hence in order to complete the proof we only need to check that F does not contain a copy of c_0.

Since $I(E)$ is an ideal in E^{**} we have from (†) that for each $f \in I(E)$ and $n = 1, 2, \ldots$,

$$\|Rf\| \leq 2^n sup\{\, \|T^{**}g\| \colon\ g \in I(E), |g| \leq |f|\,\} + 2^{-n}\|f\|. \tag{‡}$$

Now J^* is a lattice homomorphism since J is interval preserving ([AB, p. 90]). Since $T^{**}{}_{|I(E)}$ does not preserve a copy of c_0, neither does R by Lemma I.1.a of [GJ3], hence (‡) implies that F has order continuous norm by Lemma I.1.b of [82].

We first prove that if $(e_m)_{m=1}^{\infty}$ is an increasing sequence in $I(E)_+$ such that $(J^*e_m) = (Re_m)$ is norm bounded in F then it is norm convergent in F. Indeed, let μ be the w^*–limit of $(J^*e_m)_{m=1}^{\infty}$ in G^*. Since $J^*[E^{**}]$ is norm dense in G^*, there exists for each $\epsilon > 0$ an element $\nu_\epsilon \in E^{**}$ such that $\|J^*\nu_\epsilon - \mu\| \leq \epsilon$.

Since $I(E)$ is an ideal in E^{**}, the sequence $(\nu_\epsilon \wedge e_m)_{m=1}^{\infty}$ is in $I(E)$; moreover, it is increasing and norm bounded by $\|\nu_\epsilon\|$, hence $J^*(\nu_\epsilon \wedge e_m)$ norm converges to some z_ϵ in F by Theorem I.3.d in [82]. Using again the fact that J^* is a lattice homomorphism, we have that $J^*(\nu_\epsilon \wedge e_m) = J^*(\nu_\epsilon) \wedge J^*(e_m)$ hence $J^*(\nu_\epsilon \wedge e_m)$ w^*–converges in G^* to $J^*(\nu_\epsilon) \wedge \mu$. Therefore $J^*(\nu_\epsilon) \wedge \mu = z_\epsilon$ is in F. Thus we get

$$\lim_{\epsilon \to 0} \|z_\epsilon - \mu\| = \lim_{\epsilon \to 0} \|(J^*\nu_\epsilon) \wedge \mu - \mu\| \leq \lim_{\epsilon \to 0} \|J^*\nu_\epsilon - \mu\| = 0.$$

It follows that μ is in F. Since the sequence $(J^*e_m)_{m=1}^{\infty}$ is in the order interval $[0, \mu]$ and F is order continuous, it follows that $(J^*e_m)_{m=1}^{\infty}$ norm converges to μ.

Suppose now that $(f_m)_{m=1}^{\infty}$ is any increasing sequence in F. To show that it is norm convergent, it is enough to show that for any $\epsilon > 0$ there exists $(e_m)_{m=1}^{\infty}$ increasing in $I(E)_+$ such that $\|f_m - J^*e_m\| \leq \epsilon$. For that, let $z_m \in I(E)_+$ be such that $\|J^*z_m - f_m\| \leq \epsilon 2^{-m}$. Set $e_m = \vee_{i=1}^m z_i$. Then $(e_m)_{m=1}^{\infty}$ is increasing and

$$|J^*e_m - f_m| = |J^*\left(\vee_{i=1}^m z_i\right) - f_m| = |\vee_{i=1}^m J^*z_i - \vee_{i=1}^m f_i| \leq \sum_{i=1}^m |J^*z_i - f_i|.$$

It follows that

$$\|J^*e_m - f_m\| \leq \sum_{i=1}^m \|f_i - J^*z_i\| \leq \epsilon \sum_{i=1}^m 2^{-i} \leq \epsilon.$$

By the classical version of Theorem I.3 in [82], F does not contain a copy of c_0 and hence F is a weakly sequentially complete Banach lattice ([88, p.34]). ☐

In the above proof of (5) ⇒ (2), if X is Banach lattice and T is a positive operator, then every positive operator dominated by T also factors through R. Thus we get:

Corollary 2.10.2. *Let E be a Banach lattice, let T be a positive operator from E into a Banach lattice X, and assume that $T^{**}{}_{|I(E)}$ does not preserve a positive disjoint copy of c_0. Then there exists a weakly sequentially complete Banach lattice F and an interval preserving lattice homomorphism $R \colon I(E) \to F$ such that for any operator $0 \leq T_1 \leq T$ there is a positive operator $S_1 \colon F \to X$ so that $T_1^{**}{}_{|I(E)} = S_1 R$.*

Remark 16. *The examples in [84] of positive quotient mappings onto* c_0 *which do not preserve a copy of* c_0 *were shown in [85] to fail property (4) of the theorem, hence are, in Niculescu's terminology [83], operators of type B which are not of strong type B. This answers problem 4.1 in [83]. (However, if E is order continuous then* $I(E) = E$ *and so operators from E of type B are of strong type B.) These operators also give a solution to problem 4.4 in [83]; namely, there exists a lattice homomorphism T on some Banach lattice which does not preserve a copy of* c_0 *or of* l_1*, yet* T^2 *is not weakly compact, while* T^3 *must be weakly compact: see remark 2 in [82, p.164].*

Remark 17. *One of the operators mentioned in remark 1 ("*T_1*" in the terminology of [85]) maps weakly Cauchy sequences into weakly convergent sequences; this gives a counterexample to one part of Proposition 4.2 in [83].*

Remark 18. *Since a weakly sequentially complete Banach lattice has the so-called analytic Radon- Nikodym—ARN—property (cf. [89]), an operator which satisfies condition (3) of the theorem is an ARN-operator. This implication was recently proved in a direct way by Bukhvalov [90]. Although aware of [82], Bukhvalov mistakenly thought that his result could not follow from the factorizations in [82] (he applied the error from [83] mentioned in remark 2 to conclude that the operators mentioned in remark 1 satisfy condition (3) of the theorem).*

Chapter 3

Second 10^{th} Articles

3.1 Nuclear Operators on Spaces of Continuous Vector-Valued Functions

Paulette Saab and Brenda Smith

Research supported in part by an NSF Grant DMS-87500750

Abstract:
Let Ω be a compact Hausdorff space, let E be a Banach space, and let $C(\Omega, E)$ stand for the Banach space of all E-valued continuous functions on Ω under supnorm. In this paper we study when nuclear operators on $C(\Omega, E)$ spaces can be completely characterized in terms of properties of their representing vector measures. We also show that if F is a Banach space and if $T: C(\Omega, E) \to F$ is a nuclear operator, then T induces a bounded linear operator $T^{\#}$ from the space $C(\Omega)$ of scalar valued continuous functions on Ω into $\mathcal{N}(E, F)$ the space of nuclear operators from E to F, in this case we show that E^* has the Radon-Nikodym property if and only if $T^{\#}$ is nuclear whenever T is nuclear.
Primary 46E40, 46G10, 47B10, Secondary 28B05, 28B20

3.1.1 Introduction.

Let Ω be a compact Hausdorff space, let E be a Banach space, and let $C(\Omega, E)$ stand for the Banach space of continuous E-valued functions on Ω under supnorm. It is well known [91, p.182] that if F is a Banach space, then any bounded linear operator $T: C(\Omega, E) \longrightarrow F$ has a finitely additive vector measure G defined on the σ-field of Borel subsets of Ω with values in the space $\$(E, F^{**})$ of bounded linear operators from E to the second dual F^{**} of F.

The measure G is said to represent T. The purpose of this note is to study the interplay between certain properties of the operator T and properties of the representing measure G. Precisely, one of our goals is to study when one can characterize nuclear operators in terms of their representing measures. This is of course motivated by a well known Theorem of L. Schwartz [92] (see also [91, p.173]) concerning nuclear operators on spaces $C(\Omega)$ of continuous scalar-valued functions.

The study of nuclear operators on spaces $C(\Omega, E)$ of continuous vector-valued functions was initiated in [93] where the author extended Schwartz's result in case E^* has the Radon-Nikodym property. In this paper, we will show that the condition on E^* to have the Radon-Nikodym property is necessary to have a Schwartz's type theorem. This leads to a new characterization of dual spaces E^* with the Radon-Nikodym property. In [94] it was shown that if $T: C(\Omega, E) \to F$ is nuclear then its representing measure G takes its values in the space $\mathcal{N}(E, F)$ of nuclear operators from E to F. One of the results of this paper is that if $T: C(\Omega, E) \to F$ is nuclear then its representing measure G is countably additive and of bounded variation as a vector measure taking its values in $\mathcal{N}(E, F)$ equipped with the nuclear norm.

Finally, we show by easy examples that the above mentioned conditions on the representing measure G do not characterize nuclear operators on $C(\Omega, E)$ spaces, and we also look at cases where nuclear operators are indeed characterized by the above two conditions. For all undefined notions and terminologies we refer the reader to [91].

3.1.2 Preliminaries

If X and Y are Banach spaces, then $\$(X, Y)$ will stand for the space of bounded linear operators from X to Y. An element T in $\$(X, Y)$ is said to be a **nuclear operator** if there exist sequences (x_n^*) in X^* and (y_n) in Y such that for each x in X

$$T(x) = \sum_{n=1}^{\infty} x_n^*(x) y_n,$$

and

$$\sum_{n=1}^{\infty} \| x_n^* \| \| y_n \| < \infty.$$

We say that $\sum\limits_n x_n^* \otimes y_n$ represents the nuclear operator T. The **nuclear norm** of a nuclear operator $T: X \to Y$ is defined by:

$$\| T \|_{nuc} = \inf \left\{ \sum_n \| x_n^* \| \| y_n \| \right\}$$

where the infimum is taken over all sequences (x_n^*) and (y_n) such that $T(x) = \sum\limits_{n=1}^{\infty} x_n^*(x) y_n$ holds for all x in X. The nuclear operators from X to Y form a normed linear space under the nuclear norm [91, p.170], which we shall denote by $\mathcal{N}(X, Y)$. If Ω is a compact Hausdorff space and E is a Banach space, then $C(\Omega, E)$ will stand for the space of continuous E-valued functions defined on Ω under supnorm. If $E = \mathbb{R}$ or $\mathbb{C}$ we will simply write $C(\Omega)$. The space $M(\Omega, E^*)$ will stand for the space of all regular E^*-valued vector measures μ defined on the σ-field $\sum$ of Borel subsets of Ω that are of bounded variation. We shall use the fact (see [3, p.182]) that $M(\Omega, E^*)$ is a Banach space under the variation norm $\| \mu \| = |\mu|(\Omega)$, and that $M(\Omega, E^*)$ is isometrically isomorphic to the dual space $C(\Omega, E)^*$. When $E = \mathbb{R}$ or $\mathbb{C}$ we will simply write $M(\Omega)$. If $\mu \in M(\Omega, E^*)$, then for each $e \in E$ we will denote by $\langle e, \mu \rangle$ the element of $M(\Omega)$ such that for each $f \in C(\Omega)$,

$$\int f d\langle e, \mu \rangle = \mu(f \otimes e)$$

where $f \otimes e$ is the element in $C(\Omega, E)$ such that $f \otimes e(\omega) = f(\omega)e$ for each $\omega \in \Omega$.

If $\nu \in M(\Omega)$ and $x^* \in E^*$, we denote by $\nu \otimes x^*$ the element of $M(\Omega, E^*)$ which to each Borel subset B of Ω associates the element $\nu(B)x^*$ of E^*. If E and F are Banach spaces and Ω

is a compact Hausdorff space, then we will denote by G the finitely additive $\$(E, F^{**})$-valued measure representing the operator T. Recall that if B is a Borel subset of Ω then

$$G(B)e = T^{**}(\phi_{B,e})$$

for all $e \in E$, where $\phi_{B,e}$ is the element of $M(\Omega, E^*)^*$ such for each $\lambda \in M(\Omega, E^*)$

$$\phi_{B,e}(\lambda) = \lambda(B)(e)$$

and T^{**} is the second adjoint of T.

Finally we recall that a Banach space X has the **Radon-Nikodym property** (RNP) if for every finite measure space (S, Σ, μ) and every vector measure $m : \ \Sigma \to X$ of bounded variation that is absolutely continuous with respect to μ there exists a strongly measurable Bochner integrable function $g : \ S \to X$ such that

$$m(A) = \int_A f d\mu$$

for each $A \in \Sigma$.

3.1.3 Some Properties of the Measure representing a Nuclear operator

Throughout we let Ω be a compact Hausdorff space with $\sum$ its σ-field of Borel subsets and we let E and F be Banach spaces. In what follows we shall look at some of the properties that a nuclear operator on $C(\Omega, E)$ induces on its representing measure G. In [2] it was shown that if $T : \ C(\Omega, E) \to F$ is nuclear then $G : \ \sum \to \mathcal{N}(E, F)$. In the next proposition we shall show that G enjoys a stronger property.

Proposition 3.1.1. *If $T : \ C(\Omega, E) \to F$ is a nuclear operator with representing measure G, then:*

1. *for each Borel subset B of Ω, $G(B) : \ E \to F$ is a nuclear operator, and*
2. *the measure G is countably additive and is of bounded variation as a vector measure taking its values in $\mathcal{N}(E, F)$ under the nuclear norm.*

Proof. If $T : \ C(\Omega, E) \to F$ is nuclear, then there are sequences (μ_n) in $M(\Omega, E^*)$ and (y_n) in F such that for all $f \in C(\Omega, E)$

$$Tf = \sum_{n=1}^{\infty} \mu_n(f) y_n$$

and

$$\sum_{n=1}^{\infty} \parallel \mu_n \parallel \parallel y_n \parallel < \infty.$$

In particular the operator T^{**} is nuclear with

$$T^{**}(\phi) = \sum_{n=1}^{\infty} \phi(\mu_n) y_n$$

for all $\phi \in C(\Omega, E)^{**}$. This implies that for any Borel subset B of Ω and for all $e \in E$

$$G(B)e = T^{**}(\phi_{B,e})$$

$$= \sum_{n=1}^{\infty} \mu_n(B)(e) y_n$$

Since $(\mu_n(B))$ is a sequence in E^* and $\sum\limits_{n=1}^{\infty} \parallel \mu_n(B) \parallel \parallel y_n \parallel \leq \sum\limits_{n=1}^{\infty} \parallel \mu_n \parallel \parallel y_n \parallel < \infty$, this will quickly show that for each Borel subset B of Ω, the operator $G(B) : E \to F$ is nuclear. To prove (ii) note that since $G : \sum \to \mathcal{N}(E, F)$, let $|G|_{nuc}$ denote the extended non negative function whose value on a set B in $\sum$ is given by

$$|G|_{nuc}(B) = \sup_{\pi} \sum_{B_i \in \pi} \parallel G(B_i) \parallel_{nuc}$$

where the supremum is taken over all finite partitions π of B. We first show that $|G|_{nuc}(\Omega) < \infty$.

For this, note that if $\pi = \{B_i\}$ is a finite partition of Ω, then

$$\begin{aligned}
\sum_{B_i \in \pi} \parallel G(B_i) \parallel_{nuc} &\leq \sum_{B_i \in \pi} \sum_{n=1}^{\infty} \parallel \mu_n(B_i) \parallel \parallel y_n \parallel \\
&\leq \sum_{B_i \in \pi} \sum_{n=1}^{\infty} |\mu_n|(B_i) \parallel y_n \parallel \\
&\leq \sum_{n=1}^{\infty} \sum_{B_i \in \pi} |\mu_n|(B_i) \parallel y_n \parallel \\
&\leq \sum_{n=1}^{\infty} |\mu_n|(\Omega) \parallel y_n \parallel \\
&= \sum_{n=1}^{\infty} \parallel \mu_n \parallel \parallel y_n \parallel \quad < \infty
\end{aligned}$$

This of course shows that $|G|_{nuc}(\Omega) < \infty$. To complete the proof of (ii) we need to show that G is countably additive. First note that since $|G|_{nuc}(\Omega) < \infty$, it follows from [91, p.7] that G is strongly additive, that is if (B_i) is a sequence of pairwise disjoint Borel subsets of Ω we have that the series $\sum\limits_{i=1}^{\infty} G(B_i)$ converges in $\mathcal{N}(E, F)$. To complete the proof we need to check that the series $\sum\limits_{i=1}^{\infty} G(B_i)$ converges to $G(\bigcup\limits_{i \geq 1} B_i)$ in $\mathcal{N}(E, F)$. To this end, consider a series $\sum\limits_{n=1}^{\infty} \mu_n \otimes y_n$ representing the operator T such that

$$\sum_{n=1}^{\infty} ||\mu_n|| \, ||y_n|| < \infty.$$

Without loss of generality we may and shall assume that $||y_n|| \leq 1$ for all $n \geq 1$. Let $\epsilon > 0$ and pick $N \in \mathbb{N}$ such that

$$\sum_{\nu=N+1}^{\infty} ||\mu_n|| < \epsilon/2$$

Since $\mu_1, \mu_2, \ldots$, and $\mu_N \in M(\Omega, E^*)$, there exists $K \in \mathbb{N}$ such that

$$|\mu_j| \left(\bigcup_{i \geq K} B_i \right) < \frac{\epsilon}{2^{j+1}} \text{ for all } j = 1, \ldots, N$$

This implies that

$$\begin{aligned}
&||G(\bigcup_{i\geq 1} B_i) - \sum_{i=1}^{K-1} G(B_i)||_{nuc} = \\
&||G\left(\bigcup_{i\geq K} B_i\right)||_{nuc} \leq \\
&\sum_{n=1}^{\infty} ||\mu_n\left(\bigcup_{i\geq K} B_i\right)|| \leq \\
&\sum_{n=1}^{N} |\mu_n|\left(\bigcup_{i\geq K} B_i\right) + \sum_{n=N+1}^{\infty} |\mu_n|\left(\bigcup_{i\geq K} B_i\right) < \epsilon
\end{aligned}$$

This shows that G is countably additive as a vector measure taking its values in $\mathcal{N}(E,F)$. □

The first question that arises at this stage, is when do properties (i) and (ii) above characterize nuclear operators? The next proposition shows that one necessary condition is that F should have the Radon-Nikodym property.

Proposition 3.1.2. *If F fails RNP, then for any Banach space E there is a non nuclear operator $T: C([0,1],E) \to F$ whose representing measure takes its values in $\mathcal{N}(E,F)$ and is of bounded variation as a countably additive vector measure taking its values in $\mathcal{N}(E,F)$.*

Proof. If F lacks RNP then by there is[91, p.175] an F-valued countably additive vector measure m on the Borel subsets of [0,1] such that m is of bounded variation, m is absolutely continuous with respect to Lebesgue measure but m admits no Bochner integrable derivative with respect to its variation $|m|$. If we define $T': C[0,1] \to F$ by

$$T'(f) = \int_{[0,1]} f\,dm$$

then T' is not a nuclear operator (see [92] or [91, p.177]). Now fix $e \neq 0$ in E then choose e^* in E^* with $e^*(e) = 1$ and define $T: C([0,1],E) \to F$ by

$$T(\varphi) = T'(e^* \circ \varphi)$$

for each φ in $C([0,1],E)$. In particular, for each f in $C[0,1]$ and x in E we have

$$T(f \otimes x) = e^*(x)T'(f).$$

It is clear that the measure representing the operator T is the measure $G = m \otimes e^*$ which to every Borel subset B of [0,1] associates the one-rank operator such that $G(B)x = e^*(x)m(B)$ for each $x \in E$. On other hands for any finite partition $\pi = \{B_i\}$ of [0,1] we have

$$\begin{aligned}
&\sum_{B_i\in\pi} \parallel G(B_i) \parallel_{nuc} \leq \sum_{B_i\in\pi} \parallel e^* \parallel\parallel m(B_i) \parallel \\
&\leq \parallel e^* \parallel \ |m|([0,1]) < \infty \\
&|G|_{nuc}([0,1]) < \infty.
\end{aligned}$$

If T were a nuclear operator then there would exist (μ_n) in $M([0,1],E^*)$ and (y_n) in F such that for each ϕ in $C([0,1],E)$

$$T(\phi) = \sum_{n=1}^{\infty} \mu_n(\phi) y_n$$

In particular for each $f \in C[0,1]$

$$\begin{aligned} T(f \otimes e) &= T'(f) \\ &= \sum_{n=1}^{\infty} \mu_n(f \otimes e) y_n \\ &= \sum_{n=1}^{\infty} \int_{[0,1]} f d\langle e, \mu_n \rangle y_n \end{aligned}$$

This of course shows that T' is represented by $\sum\limits_n \langle e, \mu_n \rangle \otimes y_n$; moreover

$$\sum_{n=1}^{\infty} \| \langle e, \mu_n \rangle \| \| y_n \| \leq \| e \| \sum_n \| \mu_n \| \| y_n \| < \infty$$

which implies that T' is nuclear. This contradiction finishes the proof. □

This brings us to ask the following question. Let E and F be Banach spaces such that F has the Radon-Nikodym property. Let Ω be a compact Hausdorff space and let $T: C(\Omega, E) \to F$ be a bounded linear operator satisfying conditions (i) and (ii) of Proposition 2. Is T nuclear? Recently, the first named author has given a positive answer to the above question when F is complemented in its bidual F^{**} (see [95]).

3.1.4 The Radon-Nikodym Property and Nuclear Operators

Throughout this section Ω is a compact Hausdorff space and E and F are Banach spaces. Every bounded linear operator $T: C(\Omega, E) \to F$ induces a bounded linear operator $T^{\#}: C(\Omega) \longrightarrow \$(E,F)$ where for each f in $C(\Omega)$

$$T^{\#}(f)(e) = T(f \otimes e)$$

for all $e \in E$.

In this section we shall look at the interplay of the two operators T and $T^{\#}$. The next result shows that when T is nuclear, then the range of $T^{\#}$ is in $\mathcal{N}(E,F)$.

Theorem 3.1.3. *If $T: C(\Omega, E) \to F$ is nuclear, then $T^{\#}$ takes its values in $\mathcal{N}(E,F)$.*

Proof. Let (μ_n) in $M(\Omega, E^*)$ and (y_n) in F be such that $\sum\limits_{n=1}^{\infty} \| \mu_n \| \| y_n \| < \infty$ and for each φ in $C(\Omega, E)$

$$T(\varphi) = \sum_{n=1}^{\infty} \mu_n(\varphi) y_n.$$

For each $n \geq 1$ define $\mu_n^{\#}: C(\Omega) \to E^*$ by

$$\mu_n^{\#}(f)e = \mu_n(f \otimes e)$$

for each $f \in C(\Omega)$ and $e \in E$. Since for each $f \in C(\Omega)$ and $e \in E$

$$\begin{aligned}T^{\#}(f)(e) &= T(f \otimes e)\\ &= \sum_{n=1}^{\infty} \mu_n(f \otimes e) y_n\\ &= \sum_{n=1}^{\infty} \mu_n^{\#}(f)(e) y_n\end{aligned}$$

it follows that for each f in $C(\Omega)$ $T^{\#}(f)$ can be represented by the series $\sum_{n=1}^{\infty} \mu_n^{\#}(f) \otimes y_n$. Moreover since $\sum_{n=1}^{\infty} \| \mu_n^{\#}(f) \| \| y_n \| \leq \| f \| \sum_{n=1}^{\infty} \| \mu_n \| \| f_n \| < \infty$, it follows that $T^{\#}(f)$ is a nuclear operator from E to F for each $f \in C(\Omega)$. □

The next result illustrates one key relationship between T and $T^{\#}$.

Theorem 3.1.4. Theorem 4 *The operator T $C(\Omega, E) \to F$ is nuclear whenever $T^{\#}: C(\Omega) \to \mathcal{N}(E, F)$ is nuclear.*

Proof. Assume $T^{\#}: C(\Omega) \to \mathcal{N}(E, F)$ is nuclear. Then there exist sequences (ν_n) in $C(\Omega)^*$ and (N_n) in $\mathcal{N}(E, F)$ such that for each $f \in C(\Omega)$

$$T^{\#}(f) = \sum_{n=1}^{\infty} \nu_n(f) N_n$$

and

$$\sum_{n=1}^{\infty} \| \nu_n \| \| N_n \|_{nuc} < \infty.$$

Without loss of generality we may and do assume that $\| \nu_n \| \leq 1$ for all $n \geq 1$. Similarly, since each N_n is a nuclear operator, for each $n \geq 1$, there are sequences $(e^*_{n,m})$ in E^* and $(y_{n,m})$ in F such that for all $e \in E$

$$N_n(e) = \sum_{m=1}^{\infty} e^*_{n,m}(e) y_{n,m}$$

and

$$\sum_{m=1}^{\infty} \| e^*_{n,m} \| \| y_{n,m} \| \leq \| N_n \|_{nuc} + 1/2^n.$$

Hence

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \| \nu_n \| \| e^*_{n,m} \| \| y_{n,m} \| \leq \sum_{n=1}^{\infty} \| \nu_n \| \| N_n \|_{nuc} + \sum_{n=1}^{\infty} \| \nu_n \| / 2^n < \infty$$

Moreover, for each $f \in C(\Omega)$ and $e \in E$

$$\begin{aligned}T(f \otimes e) &= T^{\#}(f)(e)\\ &= \left(\sum_{n=1}^{\infty} \nu_n(f) N_n \right)(e)\\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \nu_n(f) e^*_{n,m}(e) y_{n,m}\end{aligned}$$

$$= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \nu_n \otimes e^*_{n,m}(f \otimes e) y_{n,m}$$

Since the set $\{f \otimes e : f \in C(\Omega), e \in E\}$ is total in $C(\Omega, E)$, we can assert that T is represented by the double indexed series $\sum\limits_n \sum\limits_m (\nu_n \otimes e^*_{n,m}) \otimes y_{n,m}$, where $(\nu_n \otimes e^*_{n,m})$ is in $M(\Omega, E^*) \simeq C(\Omega, E)^*$ and $(y_{n,m})$ is in F. An appeal to (††) shows that T is nuclear. □

In the following we will show that the converse of Theorem 4 does not always hold. But if in addition E^* is assumed to have the Radon-Nikodym property then a close look at [[93], Theorem III.4] reveals that any nuclear operator $T : C(\Omega, E) \to F$ will indeed induce a nuclear operator $T^\# : C(\Omega) \to \mathcal{N}(E, F)$. Moreover our next result shows how critical the condition on E^* to have the Radon-Nikodym property in order that T nuclear implies $T^\#$ nuclear. As a matter of fact one can characterize dual Banach spaces with the Radon-Nikodym property as follows:

Theorem 3.1.5. *Let E and F be Banach spaces. The following properties are equivalent:*

1. *The dual space E^* has (RNP);*
2. *For every compact space Ω, a bounded linear operator $T : C(\Omega, E) \to F$ is nuclear if and only if $T^\# : C(\Omega) \to (E, F)$ is nuclear.*

Proof. (ii)⇒(i).

If E^* fails RNP, by [91, p.175] there exists $\mu \in M([0,1], E^*)$ such that the operator $\mu^\# : C[0,1] \to E^*$ defined by

$$\mu^\#(g)e = \mu(g \otimes e)$$

for all $g \in C[0,1]$ and $e \in E$, is not nuclear. Choose $y \in F$ such that $y \neq 0$, and define $T : C([0,1], E) \to F$ by

$$T(\varphi) = \mu(\varphi)y$$

for all $\varphi \in C([0,1], E)$. It is clear that T is a rank-one operator, hence it is nuclear. The operator $T^\# : C[0,1] \to \mathcal{N}(E, F)$ induced by T is clearly the operator such that for each $g \in C[0,1]$

$$T^\#(g) = \mu^\#(g) \otimes y.$$

To see that $T^\#$ is not nuclear, note that for each $y^* \in F^*$, we can define the operator $T^\#_{y^*} : C[0,1] \to E^*$ by

$$T^\#_{y^*}(g) = (T^\# g)^*(y^*)$$

for each $g \in C[0,1]$.

If $T^\#$ were a nuclear operator then $T^\#_{y^*}$ would also be a nuclear operator for each $y^* \in F^*$. This of course follows from the fact that $T^\#_{y^*}$ is the composition of $T^\#$ and the bounded linear operator from $\mathcal{N}(E, F)$ to E^* which to an element N in $\mathcal{N}(E, F)$ associates the element $N^*(y^*)$ in E^*. But for each g in $C[0,1]$

$$T^\#_{y^*}(g) = y^*(y)\mu^\#(g).$$

By the Hahn-Banach Theorem, choose y^* in F^* such that $y^*(y) = 1$, then for this particular y^* we have

$$T^\#_{y^*} = \mu^\#.$$

This contradiction shows that $T^{\#}$ can not be nuclear. This proves (ii)⇒(i).

The proof of (i)⇒ (ii) is implicit in [1, Theorem III.4]. We shall provide a sketch of a proof for the sake of completeness. For this, assume that $T: \ C(\Omega, E) \rightarrow F$ is nuclear and that E^*has the Radon-Nikodym property. By Proposition 1 we know that the measure G representing the operator T is countably additive as a vector measure taking its values in $\mathcal{N}(E,F)$ and $|G|_{nuc}(\Omega) < \infty$. Here it is easy to note that it follows from general vector measure techniques [91, p.3] that $|G|_{nuc}$ is countably additive. The proof that $T^{\#}$ is nuclear now follows the proof of the scalar case as given in [91, p.173] with some minor changes. For instance, since E^* has the Radon-Nikodym property, one can proceed to produce a Bochner $|G|_{nuc}$-integrable function

$$H: \ \Omega \longrightarrow \mathcal{N}(E,F)$$

such that for each Borel subset B of Ω,

$$G(B) = \int_B H(\omega) d|G|_{nuc}(\omega).$$

and for each $f \in C(\Omega)$ and each $e \in E$

$$T(f \otimes e) = \int_\Omega f(\omega) H(\omega)(e) d|G|_{nuc}(\omega).$$

Hence for each $f \in C(\Omega)$

$$T^{\#}(f) = \int_\Omega f(\omega) H(\omega) d|G|_{nuc}(\omega)$$

Another appeal to [91, p.173] shows that $T^{\#}$ is nuclear.

□

3.2 Permutations of the Haar system

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Abstract:

General permutations acting on the Haar system are investigated. We give a necessary and sufficient condition for permutations to induce an isomorphism on dyadic BMO. Extensions of this characterization to Lipschitz spaces $A_{(\frac{1}{2}-1)}, (0 < p \leq 1)$ are obtained. When specialized to permutations which act on one level of the Haar system only, our approach leads to a short straightforward proof of a result due to E.M.Semyonov and B.Stoeckert.

Let us briefly describe the setting in which we are working. $\mathcal{D}$ denotes the set of all dyadic intervals contained in the unit interval. $\pi : \mathcal{D} \to \mathcal{D}$ denotes a permutation of the dyadic intervals. The operator induced by π is determined by the equation

$$T_\pi h_I = h_{\pi(I)}$$

where h_I denotes the L_∞-normalised Haar function supported on the dyadic intervall I. The main result of this paper treats *general* permutations on BMO and on Lipschitz spaces. The condition on π which controlles the boundedness of T_π is given in termes of the Carleson constant of collections of dyadic intervals. The proof of the general result given below is quite complicated. We start therefore by considering first a special class of permutation operators. To study these operators on L_p E.M. Semyonov introduced the parameter,

$$K = \sup\left\{ \frac{|\pi^{-1}(\mathcal{B})^*|}{|\mathcal{B}^*|} : \mathcal{B} \subseteq \mathcal{D} \right\}$$

where for example $\mathcal{B}^*$ denotes the pointset coverd by the collection $\mathcal{B}$
E.M Semyonov and B.Stöckert proved the following result.

Theorem 3.2.1. *If for every $I \in \mathcal{D}$ we have $|\pi(I)| = |I|$ and $K < \infty$ then for $2 \leq p < \infty$ the operator T_π is bounded on L_p*

We will obtain this result from

Theorem 3.2.2. *If for every $I \in \mathcal{D}$ we have $|\pi(I)| = |I|$ and $K < \infty$ then the operator T_π is bounded on dyadic-BMO*

Proof. Recall that for formal series $f = \sum_{I\in\mathcal{D}} a_I h_I$ the dyadic - BMO norm is given by

$$\left(\sup_{\mathcal{B}} \frac{1}{|\mathcal{B}^*|} \sum_{I\in\mathcal{B}} a_I^2 |I| \right)^{\frac{1}{2}}$$

where the supremum is extended over all collections of dyadic intervals $\mathcal{B}$. We fix now $x = \sum x_I h_I$ and obtain $T_\pi x = \sum x_{\pi^{-1}(I)} h_I$. Then choose $\mathcal{B} \subseteq \mathcal{D}$ such that

$$\frac{1}{2} ||T_\pi x||^2_{BMO} \leq \frac{1}{|\mathcal{B}^*|} \sum_{\mathcal{B}} x^2_{\pi^{-1}(I)} |I|$$

By hypothesis this expression equals with

$$\frac{1}{|\mathcal{B}^*|}\sum_{\mathcal{B}} x^2_{\pi^{-1}(I)}|\pi^{-1}(I)|$$

Which equals trivially with

$$\frac{|\pi^{-1}(\mathcal{B})^*|}{|\mathcal{B}^*|}\frac{1}{|\pi^{-1}(\mathcal{B})^*|}\sum_{\mathcal{B}} x^2_{\pi^{-1}(I)}|\pi^{-1}(I)|$$

The last expression is of course bounded by $K||x||^2_{BMO}$. This finishes the proof of Theorem2.

□

Remark 19. *1) For every permutation which satisfies* $|\pi(I)| = |(I)|$ *there exist* $\mathcal{E} \subseteq \mathcal{D}$ *and* $x \in BMO$ *for which the above chain of inequalities can be reversed. Hence for such permutations the condition* $K < \infty$ is implied by *the boundedness of* T_π. *2) As* T_π *is bounded on* L_2 *we obtain from [96]* ***Corollary2 p60*** *that* T_π *is bounded on* L_p *for* $2 < p < \infty$. *We thus obtained* ***Theorem 1*** *from* ***Theorem 2*** *by interpolation.*

Up to this point we considered permutations which act on one level of the Haar system only. We now turn to arbitrary permutations.
To do this we need a scale invariant measure for the size of collections of dyadic Intervals $\mathcal{B}$: the so called Carleson condition. This notion was studied and carefully analyzed by P.W. Jones in his work on the uniform approximation property of BMO[97].

Definition 1. 1. *$\mathcal{B}$ is said to satisfy the* K-Carleson condition *if*

$$\sup_J \frac{1}{|I|}\sum_{\{J\in\mathcal{B}: J\subset I\}} |J| \le K$$

The infimum over all such K is called the Carleson constant of $\mathcal{B}$ *and will be denoted* $CC\mathcal{B}$

2. *$max\mathcal{B}$ denotes those intervals in $\mathcal{B}$ which are not contained in other intervals of $\mathcal{B}$. As dyadic intervals are nested, the collection $max\mathcal{B}$ contains only pairwise disjoint dyadic intervals.*

Theorem 3.2.3. *Let $\pi : \mathcal{D} \to \mathcal{D}$ be any permutation. The operator T_π is an isomorphism on BMO if and only if there exists $M \in \mathbf{R}^+$ such that for all $\mathcal{B} \subset \mathcal{B}$*

$$\frac{1}{M}CC\mathcal{B} \le CC\mathcal{B} \le CC\mathcal{B}M$$

We will prove Theorem 3 by decomposing $\mathcal{D}$ so as to control the norm of T_π on smaller parts. During the decomposition process we will collect additional information concerning the interaction of the small pieces. Each iteration step is based on

Lemma 3.2.4. *Let $\pi : \mathcal{D} \to \mathcal{D}$ be a permutation which satisfies the condition of Theorem 3 Then for any $D(I) \subset \mathcal{D}$ with $D(I)^* \subset I$,$K \in \mathbf{R}^+$, and $x \in BMO$ we obtain a decomposition $D(I) = G(I) \cup S(I)$*

1. *where G(I) satisfies*

$$\sum_{J\in G(I)} x_j^2|\pi(j)| \le |\pi(D(I))^*|||x||^2_{BMO}k||x||^2_{BMO}K$$

2. *and S(I) satisfies*

$$\sum_{J\in maxS(I)} \frac{|J|}{|I|} \leq \frac{M}{K}$$

Consequently for $\mathcal{N}(I) = \pi^{-1}(max\pi D(I)$ *and* $\mathcal{O}(I) = (\mathcal{N}(I) \cap S(I)) \cup maxS(I)$ *we obtain*

$$\sum_{J\in\mathcal{O}(I)} \frac{|J|}{|I|} \leq \frac{M(M+1)}{K}$$

Proof. We define

$$S(I) = \left\{ J \in D(I) : \frac{|\pi J|}{|\pi D(I)^*|} \geq K\frac{|J|}{|I|} \right\}$$

and let $G(I) = D(I) \setminus S(I)$. The defining inequality for $J \in G(I)$ implies that

$$\sum_{J\in G(I)} x_J^2 |\pi J| \leq \sum_{J\in G(I)} x_J^2 \frac{|J|}{|I|} K|\pi D(I)^*|$$

This expression has $||x||_{BMO}^2 K|\pi D(I)^*|$ as upper bound. It remains to analyze S(I): by definition of S(I) we get

$$\sum_{J\in maxS(I)} \frac{|J|}{|I|} \leq \frac{1}{K} \sum_{J\in maxS(I)} \frac{|\pi J|}{|\pi D(I)^*|}$$

The sum on the right hand side is dominated by the Carleson constant of $\pi maxS(I)$, which by assumtion is bounded by M times the Carleson constant of $maxS(I)$. However $maxS(I)$ beeing a collection of pairwise satisfies the 1 Carleson condition. Summing up we obtain

$$\sum_{J\in maxS(I)} \frac{|J|}{|I|} \leq \frac{M}{K}$$

The collection $max\pi D(I)$ like any other collection of disjoint dyadic intervals has Carleson constant equal to one. Hence $\pi^{-1}(max\pi D(I))$ satisfies the M Carleson condition. In particular for $L \in maxS(I)$

$$\sum_{\{J\in\mathcal{N}(I) : J\subset L\}} |J| \leq M|L|$$

Let us combine this estimate with the previous analysis of S(I) to describe the size of $\mathcal{O}(I)$.

$$\sum_{J\in\mathcal{O}(I)} |J| \leq \sum_{L\in maxS(I)} \sum_{\{J\in\mathcal{N}(I) : J\subset L\}} |J| \leq \sum_{L\in maxS(I)} M|L| \leq \frac{M^2}{K}|I|$$

The generations of the index set $\mathcal{O}(I)$ are used to form a stopping time decomposition of S(I). ☐

Definition 2. 1. *We first recall how generations are formed: let* $G_0(\mathcal{O}(I))$ *be* $max\mathcal{O}(I)$. *Having defined* $G_0(\mathcal{O}(I)), \ldots, G_l(\mathcal{O}(I))$ *we put*

$$G_{l+1}(\mathcal{O}(I)) = max\left(\mathcal{O}(I) \setminus \bigcup_{k\leq l} G_k(\mathcal{O}(I))\right)$$

2. *We now form the crucial decomposition of S(I):*

   For $k \in \{0, 1, 2, \ldots\}$ *and* $L \in G_k(\mathcal{O}(I))$ *we define* $D(L) = \{J \in S(I) : J \subset L\} \setminus \{J \subset P : P \in G_{k+1}(\mathcal{O}(I))\}$

Comment

1) Consider the following identity

$$\sum_{J\in D(I)} x_J^2|\pi(J)| = \sum_{J\in G(I)} x_J^2|\pi(J)| + \sum_{L\in\mathcal{O}(I)}\sum_{J\in D(L)} x_J^2|\pi(J)|$$

In view of Lemma1 the sum indexed by G(I) admits a good upper bound. We shifted the bad behavior of the permutation into the sum indexed by $\mathcal{O}(I)$ However we used the hypothesis to show that this index set is geometrically small compared to I. This remark indicates that Lemma1 permits us to show that a repeated application of the identity defines a converging algorithm.
2) One's first idea might be to choose maxS(I) as index set $\mathcal{O}(I)$. However when one tries to prove convergence for the associated decomposition procedure one meets serious technical difficulties. The way to get around these complications is to choose an index set which contains information about $maxS(I)$ **and** $\pi^{-1}(max\pi(D(I))$.

Proof of Theorem3. The necessity of our condition is implied by the following relation between the Carleson condition and BMO : $CC\mathcal{B} = ||\sum_{I\in\mathcal{B}} h_I||^2_{BMO}$. The rest of the paper is used to show that the condition of Theorem3 is also sufficient. We first choose $J_0 \in \mathcal{B}$ such that $\frac{1}{2}||T_\pi x||^2$ is bounded by

$$\frac{1}{|\pi(J_0)|}\sum_{\{J:\pi(J)\subset\pi(J_0)\}} x_J^2|\pi(J)|$$

We now let $\mathcal{B} = \pi^{-1}(\{J : \pi(J) \subset \pi(J_0)\})$ and $\mathcal{O}_0 = max\mathcal{B}$ For $I \in \mathcal{O}_0$ we put $D(I) = \{J \in \mathcal{B} : J \subset I\}$ having produced $\mathcal{O}_1 \ldots \mathcal{O}_l, \mathcal{B}_1 \ldots \mathcal{B}_l$ and $\mathcal{N}_1 \ldots \mathcal{N}_l$ we choose $I \in \mathcal{O}_l$ and $D(I) \in \mathcal{B}_l$. Lemma1 is now applied to D(I) and we obtain G(I), S(I), $\mathcal{N}(I)$, and $\mathcal{O}(I)$. Finally S(I) is decomposed according to the Definition2 . This gives us $\{D(L) \subset S(I) : L \in \mathcal{O}(I)\}$. Doing this for each $I \in \mathcal{O}_l$ allowes us to define

$$\mathcal{O}_{l+1} = \bigcup_{I\in\mathcal{O}_l}\mathcal{O}(I), \mathcal{B}_{l+1} = \bigcup_{I\in\mathcal{O}_{l+1}} D(I), \mathcal{N}_{l+1} = \bigcup_{I\in\mathcal{O}_l}\mathcal{N}(I)$$

We thereby completed the induction step. The next two claims describe the behaviour of our construction. From now on, K denotes any number bigger than $4M^2$

□

Claim 3.2.5. $\mathcal{O} = \bigcup_{l\in\mathbf{N}}\mathcal{O}_l$ *satisfies the 2 Carleson condition*

Proof. Fix $I \in \mathcal{O}_{k_0}$. If $J \in \mathcal{O}_k$, and $J \subset I$ then by construction we obtain $k \geq k_0$. Applying the estimates of Lemma1 we obtain for $k \geq k_0$

$$\sum_{\{J\in\mathcal{O}_k:J\subset I\}} |J| \leq \left(\frac{M(M+1)}{K}\right)^{k-k_0} |I|$$

For $I \in \mathcal{O}_{k_0}$

$$\sum_{\{J\in\mathcal{O}:J\subset I\}} |J| = \sum_{k\geq k_0}\sum_{\{J\in\mathcal{O}_k:J\subset I\}} |J|$$

Invoking the observation above we obtain for this sum the following majorization:

$$|I|\sum_{k\geq k_0}\left(\frac{M(M+1)}{K}\right)^{k-k_0}\leq |I|2$$

□

Claim 3.2.6. $\mathcal{N}=\bigcup_{l\in\mathbf{N}}\mathcal{N}_l$ *satisfies the (3M) Carleson condition.*

Proof. When we look back at the construction we see that each dyadic interval lies in at most one of the collections $\mathcal{N}(J)$. Fix now $I\in\mathcal{N}$. Hence $I\in\mathcal{N}_{k_0}$ for some k_0 , and there exists exactly one dyadic interval P such that $I\in\mathcal{N}(P)$. This remark gives the representation

$$\{J\in\mathcal{N}:J\subset I\}=\{J\in\mathcal{N}(P):J\subset P\}\cup\{\mathcal{N}(L):L\in\mathcal{O},L\subset P\}$$

We thus obtain the following identity:

$$\frac{1}{|I|}\sum_{\{J\in\mathcal{N}:J\subset I\}}|J|=\frac{1}{|I|}\sum_{J\in\mathcal{N}(P)}|J|+\frac{1}{|I|}\sum_{\{L\in\mathcal{O}:L\subset P\}}\sum_{\{J\in\mathcal{N}(L)\}}|J|$$

The first summand is simply estimated by $\mathcal{N}$, $\surd$ which in turn is less than M.The second term is majorized by

$$\frac{1}{|I|}\sum_{\{L\in\mathcal{O}:L\subset P\}}M|L|\leq MCC\mathcal{O}$$

Having proved claim2,**we resume the proof of Theorem3.**

We know now that $\mathcal{N}$ satisfies the 3M Carleson condition. Observe that $\pi\mathcal{N}=\bigcup_{I\in\mathcal{O}}max\pi D(I)$ Therefore

$$\frac{1}{|\pi(J_0)|}\sum_{I\in\mathcal{O}}|max\pi(D(I))^*|\leq CC\pi(\mathcal{N})$$

By hypothesis on π we get $CC\pi(\mathcal{N})\leq M.CC\mathcal{N}$. Recall next that $\{G(I):I\in\mathcal{O}\}$ is a decomposition of $\mathcal{B}$. We thus obtain the following final estimates.

$$\frac{1}{2}||T_\pi x||^2_{BMO}\leq\frac{1}{|\pi(J_0)|}\sum_{\{\pi(J):J\in\mathcal{B}\}}x_J^2|\pi(J)|=\frac{1}{|\pi(J_0)|}\sum_{I\in\mathcal{O}}\sum_{\{\pi(J):J\in G(I)\}}x_J^2|\pi(J)|$$

$$\leq\frac{1}{|\pi(J_0)|}\sum_{I\in\mathcal{O}}|max\pi(D(I)^*|||x||^2$$

As we observed above this sum is bounded from above by $3M^2||x||^2_{BM}$

□

This proves that T_π is a bounded operator on BMO. As the hypothesis of the theorem is symmetric in π and π^{-1} we conclude that $T_{\pi^{-1}}=T_\pi^{-1}$ is bounded as well.

Extensions to Lipschitz functions: The dyadic BMO condition appears as a natural limit of some Lipschitz condition for martingales. We describe subsequenty an extension of our main result to Lipschitz spaces. l For $\varphi\in L^2$ say and $0<p\leq 1$ the Lipschitz condition assumes the form

$$||\varphi||_{\Lambda_{\left(\frac{1}{p}-1\right)}}=\sup_{I\in\mathcal{D}}\left(\int_I|\varphi-\varphi_I|^2\frac{dt}{|I|}\right)^{\frac{1}{2}}|I|^{1-\frac{1}{p}}$$

where φ_I denotes the meanvalue of φ over I. The particular interest in these class of functions stems from a duality relation due to C. Herz,[**?**], which generalizes C.Feffermans duality theorem. Herz's theorem identifies $\Lambda_{\left(\frac{1}{p}-1\right)}$ with the dual space of dyadic $H^p(0 < p \leq 1)$.

We shall now discuss permutations of Haar functions which are normalised in $\Lambda_{\left(\frac{1}{p}-1\right)}$. The norm of the L^∞ normalised functions h_I in $\Lambda_{\left(\frac{1}{p}-1\right)}$ equals $|I|^{1-\frac{1}{p}}$. Hence the operator induced by the permutation $\pi : \mathcal{D} \to \mathcal{D}$ is given by the relation

$$T_{p,\pi}\left(\frac{h_I}{|I|^{1-\frac{1}{p}}}\right) = \frac{h_{\pi(I)}}{|\pi(I)|^{1-\frac{1}{p}}}$$

It is useful to observe that the $\Lambda_{\left(\frac{1}{p}-1\right)}$ norm of $f = \sum a_I \frac{h_I}{|I|^{1-\frac{1}{p}}}$ can be expressed in terms of the coefficients a_I. In fact we obtain

$$||f||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}} = \sup_{\mathcal{B}}\left(\frac{1}{|I|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}\sum_{J\subset I} a_J^2 |J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}\right)$$

This formula suggests how to extend properly the notion of Carleson - constant.

Definition 3. *For a collection $\mathcal{B}$ of dyadic intervals, the* Carleson p-constant *is given by*

$$\sup_{I\in\mathcal{B}}\left(\frac{1}{|I|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}\sum_{\{J\in\mathcal{B}:J\subseteq I\}} |J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}\right)$$

It will be denoted by $\mathrm{CC_p}(\mathcal{B})$

Our extension of Theorem 3 reads now as follows

Theorem 3.2.7. *Let $\pi : \lceil \to \lceil$ be a permutation. Let $0 < p \leq 1$. The operator $T_{p,\pi}$ is an isomorphism on $\Lambda_{\left(\frac{1}{p}-1\right)}$ if and only if there exists $M \in \mathbf{R}^+$ such that for every $\lfloor \subseteq \lceil$*

$$\frac{1}{M}\mathrm{CC_p}(\lfloor) \leq \mathrm{CC_p}(\pi(\lfloor)) \leq \mathrm{CC_p}(\lfloor)\mathcal{M}$$

The proof of theorem4 follows the pattern explained above. The first step is again given by the following

Lemma 3.2.8. *Let $\pi : \lceil \to \lceil$ be a permutation which satisfies the condition of Theorem 4 Then for any $D(I) \subset \lceil$ with $D(I)^* \subset I$,$K \in \mathbf{R}^+$, and $x \in \Lambda_{\left(\frac{1}{p}-1\right)}$ we obtain a decomposition $D(I) = G(I) \cup S(I)$*

1. *where G(I) satisfies*

$$\sum_{\{\pi J:J\in G(I)\}} x_J^2 |\pi J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \sum_{L\in max\pi(D(I))} |L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} ||x||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}} K$$

2. *and S(I) satisfies*

$$\sum_{J\in maxS(I)}\left(\frac{|J|}{|I|}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \frac{M}{K}$$

Consequently for $\mathcal{N}(I) = \pi^{-1}(max\pi D(I)$ *and* $\mathcal{O}(I) = (\mathcal{N}(I) \cap S(I)) \cup maxS(I)$ *we obtain*

$$\sum_{J \in \mathcal{O}(I)} \left(\frac{|J|}{|I|}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \frac{M(M+1)}{K}$$

Proof. We define

$$S(I) = \left\{ J \in D(I) : \frac{|\pi J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}{\sum_{L \in max\pi(D(I))} |L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}} \geq K \left(\frac{|J|}{|I|}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \right\}$$

and let $G(I) = D(I) \setminus S(I)$. The defining inequality for $J \in G(I)$ implies that

$$\sum_{J \in G(I)} x_J^2 |\pi J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \sum_{J \in G(I)} x_J^2 \left(\frac{|J|}{|I|}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \sum_{L \in max\pi(D(I))} |L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}$$

This expression has

$$||x||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}} K \sum_{L \in max\pi(D(I))} |L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}$$

as upper bound. It remains to analyze S(I): by definition of S(I) we get

$$\sum_{J \in maxS(I)} \left(\frac{|J|}{|I|}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \frac{1}{K} \sum_{J \in maxS(I)} \frac{|\pi(J)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}{\sum_{L \in max\pi(D(I))} |L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}$$

The sum on the right hand side is dominated by the Carleson p constant of $\pi maxS(I)$, which by assumtion is bounded by M times the Carleson p constant of $maxS(I)$. This in turn is bounded by one. We obtained:

$$\sum_{J \in maxS(I)} \left(\frac{|J|}{|I|}\right)^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \frac{M}{K}$$

□

The collection $max\pi D(I)$ has Carleson p constant equal to one.

Hence $\pi^{-1}(max\pi D(I))$ satisfies the M Carleson p condition. In particular for $L \in maxS(I)$

$$\sum_{\{J \in \mathcal{N}(I) : J \subset L\}} |J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq M |L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}$$

This estimate and the previous analysis of S(I) gives an upper bound for the size of $\mathcal{O}(I)$.

$$\sum_{J \in \mathcal{O}(I)} |J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \sum_{L \in maxS(I)} \sum_{\{J \in \mathcal{N}(I) : J \subset L\}} |J|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}$$

$$\leq \sum_{L \in maxS(I)} M |L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \frac{M^2}{K} |I|$$

Proof of Theorem 4. The necessity of our condition is implied by the following fact

$$\mathrm{CC_p}(1)\lfloor = ||\sum_{I\in\mathcal{B}}\frac{h_I}{|I|^{1-\frac{1}{p}}}||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}}.$$

We show now that the condition of Theorem4 is also sufficient. We first choose $J_0 \in \mathcal{D}$ such that $\frac{1}{2}||T_{p,\pi}x||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}}$ is bounded by

$$\frac{1}{|\pi(J_0)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}\sum_{\{J:\pi(J)\subset\pi(J_0)\}} x_J^2|\pi(J)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}$$

We now let $\mathcal{B} = \pi^{-1}(\{J : \pi(J) \subset \pi(J_0)\})$. Using Lemma2 we form as in the proof of Theorem3 the collections $\mathcal{O},\mathcal{N}$, $\{G(I) : I \in \mathcal{O}\}$ and $\{D(I) : I \in \mathcal{O}\}$ such that :

1. $\mathrm{CC_p}(1)\mathcal{N} \leq 3M$
2. $\{G(I) : I \in \mathcal{O}\}$ is a decomposition of $\mathcal{B}$
3. $$\sum_{J\in G(I)} x_J^2|\pi(J)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \sum_{L\in max\pi(D(I))}|L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}||x||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}}K$$

As $\pi\mathcal{N} = \bigcup_{I\in\mathcal{O}} max\pi D(I)$ we obtain

$$\frac{1}{|\pi(J_0)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}\sum_{I\in\mathcal{O}}\sum_{L\in max\pi(D(I))}|L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} \leq \mathrm{CC_p}(1)\pi(\mathcal{N})$$

By hypothesis on π we get $\mathrm{CC_p}(1)\pi(\mathcal{N}) \leq M\mathrm{CC_p}(1)\mathcal{N}$. Moreover $\{G(I) : I \in \mathcal{O}\}$ is a decomposition of $\mathcal{B}$. We thus obtain

$$\frac{1}{2}||T_{p,\pi}x||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}} \leq \frac{1}{|\pi(J_0)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}\sum_{\{\pi(J):J\in\mathcal{B}\}} x_J^2|\pi(J)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)} =$$

$$\frac{1}{|\pi(J_0)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}\sum_{I\in\mathcal{O}}\sum_{\{\pi(J):J\in G(I)\}} x_J^2|\pi(J)|$$

$$\leq \frac{1}{|\pi(J_0)|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}}\sum_{I\in\mathcal{O}}\sum_{L\in max\pi(D(I))}|L|^{2\left(\frac{1}{p}-\frac{1}{2}\right)}||x||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}}$$

We observed already that this sum is bounded by $3M^2||x||^2_{\Lambda_{\left(\frac{1}{p}-1\right)}}$ □

This proves that $T_{p,\pi}$ is a bounded operator on $\Lambda_{\left(\frac{1}{p}-1\right)}$. Repeating this procedure with π replaced by π^{-1} wwe are able to conclude that $T_{p,\pi^{-1}} = T_{p,\pi}^{-1}$ is bounded as well.

Remark

In view of the above mentioned duality results due to C. Herz and C. Fefferman the transposed operator of $T_{p,\pi}$, $(0 < p \leq 1)$ is given by the operator $S_{p,\pi}$ defined by π^{-1} acting on H^p normalised Haar functions. More precisely $S_{p,\pi} : H^p \to H^p$ is given by the relation.

$$S_{p,\pi}\left(\frac{h_I}{|I|^{\frac{1}{p}}}\right) = \frac{h_{\pi^{-1}(I)}}{|\pi^{-1}(I)|^{\frac{1}{p}}}$$

An equivalent formulation of Theorem 4 is thus given by

Theorem 3.2.9. *Let* $\pi : \mathcal{D} \to \mathcal{D}$ *be a permutation. Let* $0 < p \leq 1$*. The operator* $S_{p,\pi}$ *is an isomorphism on dyadic* H^p *if and only if there exists* $M \in \mathbf{R}^+$ *such that for every* $\mathcal{B} \subseteq \mathcal{D}$

$$\frac{1}{M}\mathrm{CC_p}(\mathcal{B}) \leq \mathrm{CC_p}(\pi(\mathcal{B})) \leq \mathrm{CC_p}(\mathcal{B})M$$

3.3 On The Complemented Subspaces of X_p

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Abstract:
In this paper we prove some results related to the problem of isomorphically classifying the complemented subspaces of X_p. We characterize the complemented subspaces of X_p which are isomorphic to X_p by showing that such a space must contain a canonical complemented subspace isomorphic to X_p. We also give some characterizations of complemented subspaces of X_p isomorphic to $\ell_p \oplus \ell_2$.

3.3.1 Introduction.

Rosenthal [98] introduced the $\mathcal{L}_p$ space X_p in 1971. Among its interesting properties are that it contains and is contained in isomorphs of $\ell_p \oplus \ell_2$, but is not isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$. These properties have made X_p rather resistant to standard approaches to classifying its complemented subspaces. For example it was first proved that X_p was primary in [99] where the device of simultaneously L_p, L_2 bounded operators was employed to prove a version of the decomposition method for X_p. The problem really is that the ℓ_p and ℓ_2 structures in X_p are mixed in a much different way than they are in $\ell_p \oplus \ell_2$ or $\left(\sum \ell_2\right)_p$. Let us also recall that X_p or really the technique for building X_p is the central device used to construct an uncountable number of separable $\mathcal{L}_p$ spaces, [100]. Thus a better understanding of X_p is critical for the study of the complemented subspaces of L_p.

In order to state precisely our results we need to introduce some special notation. Throughout this paper $w = (w_n)$ will be a sequence of positive real numbers and $2 < p < \infty$. As usual $X_{p,w}$ is the completion of $\{(a_i) : i \in \mathbb{N},\ a_i \neq 0 \text{ for finitely many } i\}$ with the norm $||(a_i)|| = \max\{|(a_i)|_p,\ |(a_i)|_{2,w}\}$ where $|(a_i)|_p = [\sum_i |a_i|^p]^{1/p}$ and $|(a_i)|_{2,w} = [\sum_i |a_i|^2 w_i^2]^{1/2}$.

The Rosenthal space X_p is $X_{p,w}$ where $w = (w_i)$ is such that for every $\epsilon > 0$, $\sum\limits_{w_i < \epsilon} w_i^{2p/(p-2)} = \infty$. Throughout this paper we will always consider X_p to be the subspace of $(\ell_p \oplus \ell_2)_\infty$ spanned by $(\delta_n + w_n \gamma_n)$ where (δ_n) and (γ_n) are the usual unit vector bases of ℓ_p and ℓ_2, respectively. If $E \subset \mathbb{N}$, the symbol $\omega(E)$ will be used to denote $\sum_{n \in E} w_n^{2p/(p-2)}$ which occurs frequently in computations in X_p. We will also need the ratio of 2-norm and p-norm, $|x|_2/|x|_p$, which we will denote by $r(x)$. For a subspace Y of X_p, define $r(Y) = \sup\{r(y) : y \in Y\}$ and $h(Y) = \inf\{r(y) : y \in Y\}$. In defining functionals on X_p it is convenient to use the inner product induced by the norm $|\cdot|_2$. Thus $< (x_n),\ (y_n) >= \sum x_n y_n w_n^2$.

We will use standard Banach space notation and terminology as may be found in [101]. Here subspace will mean infinite dimensional closed subspace unless otherwise noted. The properties of X_p can be found in [101] or in the original paper of Rosenthal [98].

3.3.2 Complemented subspaces of X_p which contain X_p complemented

In this section we will show that a complemented subspace of X_p which contains a complemented subspace isomorphic to X_p contains a canonical complemented copy of X_p. In [98] Rosenthal showed that there were nice block bases of the usual basis of X_p with complemented closed linear span. The point of his construction was to make sure that the coordinate functionals of the block basis could be chosen to be bounded in both the p and 2 norms. Explicitly, if for each $j \in \mathbb{N}$,

$$y_j = \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} e_n$$

where (e_n) is the natural basis for X_p and (e_n^*) is the corresponding sequence of biorthogonal functionals, then

$$|y_j|_2 = \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/2} \text{ and } |y_j|_p = \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/p} .$$

Let

$$y_j^*(x) = |y_j|_2^{-2} \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} w_n^2 e_n^*(x) = |y_j|_2^{-2} < y_j, x > .$$

By applying Hölder's inequality first in ℓ_2 and then in $\ell_{p/2}$ we see that

$$\begin{aligned}
|y_j^*(x)| &= |y_j|_2^{-2} \left\{ \left| \sum_{n=k_j+1}^{k_{j+1}} w_n^{2/(p-2)} w_n^2 e_n^*(x) \right| \right. \\
&\leq |y_j|_2^{-2} \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{4/(p-2)} w_n^2 \right]^{1/2} \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^2 |e_n^*(x)|^2 \right]^{1/2} \\
&\leq |y_j|_2^{-2} \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/2} \min \left\{ \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^2 |e_n^*(x)|^2 \right]^{1/2} , \right. \\
&\qquad \left. \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{(p-2)/2p} \left[\sum_{n=k_j+1}^{k_{j+1}} |e_n^*(x)|^p \right]^{1/p} \right\} \\
&= \min \left\{ |x_{|[k_j+1, k_{j+1}]}|_2 |y_j|_2^{-1}, \ |x_{|[k_j+1, k_{j+1}]}|_p |y_j|_p^{-1} \right\} .
\end{aligned}$$

Thus

$$|y_j^*(x) y_j|_2 \leq |x_{|[k_j+1, k_{j+1}]}|_2 \text{ and } |y_j^*(x) y_j|_p \leq |x_{|[k_j+1, k_{j+1}]}|_p .$$

The important point is that this computation works because

$$\begin{aligned}
|y_j|_2 &= \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{(p-2)/2p} \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{1/p} \\
&= \left[\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)} \right]^{(p-2)/2p} |y_j|_p .
\end{aligned}$$

Remark 20. *The choice of coefficients for y_j has the following geometric motivation. These coefficients give the maximum for the ratio $|\cdot|_2/|\cdot|_p$ for elements supported on $[k_j+1, \ k_{j+1}]$. Thus if x is supported on $[k_j+1, \ k_{j+1}]$ and $x \in B_{\ell_p}$ (the unit ball of ℓ_p) then $(|y_j|_p/|y_j|_2)x \in B_{|\cdot|_2}$ and there is exactly one x so that the multiple has 2-norm one.*

If we replace y_j by any z_j with the same support which satisfies

$$c|z_j|_2 \geq [\sum_{n=k_j+1}^{k_{j+1}} w_n^{2p/(p-2)}]^{(p-2)/2p}|z_j|_p = \omega([k_j+1,k_{j+1}])^{p-2)/2p}|z_j|_p$$

and define

$$z_j^*(x) = |z_j|_2^{-2} < z_j, x >= |z_j|_2^{-2} \sum_{n=k_j+1}^{k_{j+1}} e_n^*(z_j)e_n^*(x)w_n^2.$$

applying Hölder's inequality as above shows that

$$|z_j^*(x)z_j|_2 \leq |x_{|[k_j+1,k_{j+1}]}|_2 \text{ and } |z_j^*(x)z_j|_p \leq c|x_{|[k_j+1,k_{j+1}]}|_p.$$

Combining this observation with the isomorphic classification of subspaces spanned by block basic sequences, we arrive at a prototype for a complemented subspace of X_p isomorphic to $X_{p,w'}$.

Proposition 3.3.1. *If (z_j) is a normalized block basis of the natural basis of X_p, (E_j) is a sequence of disjoint subsets of $\mathbb{N}$ and there are positive constants c and δ such that for all j*

1. $|z_{j|E_j}|_2 \geq \delta|z_j|_2$
2. $c|z_{j|E_j}|_2 \geq \omega(E_j)^{(p-2)/2p}$

Then $[z_j]$ is a $\max\{\delta^{-1}, c\}$ *complemented subspace of X_p isomorphic to $X_{p,w'}$, where $w'_j = \omega(E_j)^{(p-2)/2p}$.*

Proof. A block basis of the natural basis of X_p spans a subspace isomorphic to $X_{p,w''}$ where $w''_j = |z_j|_2/|z_j|_p = r(z_j)$. By Hölder's inequality and b),

$$|z_j|_p\,\omega(E_j)^{(p-2)/2p} \geq |z_{j|E_j}|_p\,\omega(E_j)^{(p-2)/2p} \geq |z_{j|E_j}|_2 \geq \delta|z_j|_2.$$

Hence

$$\delta^{-1}\omega(E_j)^{(p-2)/2p} \geq r(z_j) \geq c^{-1}\omega(E_j)^{(p-2)/2p}$$

and thus $X_{p,w'}$ is isomorphic to $X_{p,w''}$. Define a projection onto $[z_j : j \in \mathbb{N}]$ by

Clearly P is the required operator if it is bounded. The computations above using b) in the form

$$(c|z_{j|E_j}|_p)|z_{j|E_j}|_2 \geq \omega(E_j)^{(p-2)/2p}|z_{j|E_j}|_p$$

show that.

$$||\mathrm{Px}|| \leq \max\{\left[\textstyle\sum_{j=1}^{\infty}|z_j^*(x)z_j|_2^2\right]^{1/2}, \left[\textstyle\sum_{j=1}^{\infty}|z_j^*(x)z_j|_p^p\right]^{1/p}\}$$

$$= \max\left\{\left[\textstyle\sum_{j=1}^{\infty}|z_j^*(x)z_{j|E_j}|_2^2|z_j|_2^2/|z_{j|E_j}|_2^2\right]^{1/2}, \left[\textstyle\sum_{j=1}^{\infty}|z_j^*(x)z_{j|E_j}|_p^p|z_j|_p^p/|z_{j|E_j}|_p^p\right]^{1/p}\right\}$$

$$\leq \max\left\{\left[\textstyle\sum_{j=1}^{\infty}|x_{|E_j}|_2^2\delta^{-2}\right]^{1/2}, \left[\textstyle\sum_{j=1}^{\infty}|x_{|E_j}|_p^p c^p\right]^{1/p}\right\} \leq \max\{\delta^{-1}, c\}||x||. \quad \square$$

□

Next we will prove our characterization of the complemented subspaces of X_p which contain X_p complemented and thus are isomorphic to X_p, [**?**].

Theorem 3.3.2. *Suppose that X is a complemented subspace of X_p. Then the following are equivalent.*

1. *X contains a complemented subspace isomorphic to X_p*
2. *There exist positive constants c and δ such that for every $\epsilon > 0$ there is an ϵ', $0 < \epsilon' < \epsilon$, such that for every $N \in \mathbb{N}$ there is an $x \in X$, $||x|| = 1$ and a finite set $E \subset \{N, N+1, \dots\}$ such that*
3. *$||x_{|[1,N]}|| < N^{-1}$*
4. *$|x_{|E}|_2 \geq \delta|x|_2$*
5. *$\epsilon \geq c|x_{|E}|_2 \geq \omega(E)^{(p-2)/2p} \geq \epsilon'$.*

Proof. Suppose that 2) is satisfied. Let $\epsilon_k = k^{-1}$. By induction we may choose for each k a sequence $(x_{k,j})$ of norm one elements of X which are a perturbation of a block basis of the basis of X_p satisfying b) and c) (for ϵ_k' and $E_{k,j}$). Clearly we may assume that $E_{k,j} \cap E_{k,m} = \phi$ for $j \neq m$. By a simple diagonalization argument we can find sets $\mathcal{F}_k \subset \mathbb{N}$ such that $(x_{k,j})_{k=1,\, j\in\mathcal{F}_k}^{\infty}$ is equivalent to a block basis of X_p, the sets $E_{k,j},\ k \in \mathbb{N},\ j \in \mathcal{F}_k$ are disjoint, and $(\epsilon_k')^{2p/(p-2)}\text{card } \mathcal{F}_k \geq 1$ for each k. It now follows from Proposition 1.2 and standard perturbation arguments that $Y = [x_{k,j} : k \in \mathbb{N}, j \in \mathcal{F}_k]$ is isomorphic to X_p and that Y is complemented in X_p.

For the converse we will actually show that if we take as our isomorph of X_p the special representation $X_{p,w'}$, where w' is actually a doubly indexed sequence $w' = (w_{k,j})$, where $w_{k,j} = w_k$ for all j, $\lim w_k = 0$, and $\sum_{k=1}^{\infty} w_k^{2p/(p-2)} = \infty$, then the images of a subsequence of the basis satisfy the properties in 2). Thus we suppose that Y is a complemented subspace of X and that T is an isomorphism of $X_{p,w'}$ onto Y. By passing to a subsequence of the basis of $X_{p,w'}$ and using a standard perturbation argument, we may assume that Y is the span of a block of the basis of the containing X_p. Let (y_i) be the normalized basis of Y and let F_i be the support of y_i relative to the basis of X_p.

Let y_i^* denote the biorthogonal functional to y_i. Because Y is complemented in X_p, we may assume that each y_i^* is defined on X_p and $\sup ||y_i^*|| \leq ||T^{-1}||\, ||Q||$ where Q is the projection onto Y. Because Y is reflexive, $y_i^*(e_j) \to 0$ as $i \to \infty$ for each j where e_j denotes the jth basis vector of X_p. Thus we may assume by passing to a subsequence, a perturbation argument and perhaps enlarging the sets F_i slightly that $y_i^*(x) \neq 0$ only if $x_{|F_i} \neq 0$. In other words $y_i^*(x) = y_i^*(x_{|F_i})$. Also it follows from this that the projection Q onto Y is given by $Qx = \sum_{i=1}^{\infty} y_i^*(x) y_i$. Fix i and let $E_i = \{j \in F_i : |y_i(j)| \geq \rho w_j^{2/(p-2)} |y_i|_2^{-2/(p-2)}\}$ and assume that b) is satisfied for y_i and E_i.

$$
\begin{aligned}
\omega(E_i) &= \sum_{j \in E_i} w_j^{2p/(p-2)} \leq \rho^{-2} \sum_{j \in E_i} |y_i(j)|^2 |y_i|_2^{4/(p-2)} w_j^2 \\
&= \rho^{-2} |y_{i|E_i}|_2^2 |y_i|_2^{4/(p-2)} \\
&\leq \rho^{-2} \delta^{-4/(p-2)} |y_{i|E_i}|_2^{2p/(p-2)}.
\end{aligned}
$$

Thus if ρ is independent of i, condition b) will imply the middle inequality in condition c) (with $c = \rho^{-(p-2)/p} \delta^{-2/p}$). Also observe that because

$$
|y_i|_p \, \omega(E_i)^{(p-2)/2p} \geq |y_{i|E_i}|_p \, \omega(E_i)^{(p-2)/2p} \geq |y_{i_{|E_i}}|_2 \geq \delta |y_i|_2
$$

the third inequality in c) will be satisfied if $|y_i|_2$ is bounded away from zero. Hence it is sufficient to show that for some $\rho > 0$ there is a δ, $0 < \delta < 1$, such that if $\mathcal{E}_\delta = \{i : |y_{i|E_i}|_2 \geq \delta |y_i|_2\}$ then for every $\epsilon_1 > 0$ there is an $\epsilon_2 > 0$ such that $\epsilon_1 \geq |y_i|_2 \geq \epsilon_2$ for infinitely many i in $\mathcal{E}_\delta$. Then c) will be satisfied with $c\epsilon_1 = \epsilon$ and $\epsilon' = \delta \epsilon_2$.

Note that $\sum_{j \notin E_i} |y_i(j)|^p \leq \sum_{j \notin E_i} |y_i(j)|^2 \rho^{p-2} w_j^2 |y_i|_2^{-2} \leq \rho^{p-2}$. Hence $||y_{i|E_i}|| \geq [1 - \rho^{p-2}]^{1/p}$ and $|y_{i|F_i \setminus E_i}|_p \leq \rho^{1-2/p}$. Thus if ρ is small, $[y_{i|E_i^c} : i \notin \mathcal{E}_\delta]$ is not better than $\rho^{-1+2/p}$ equivalent to $[y_i : i \notin \mathcal{E}_\delta]$.

For each K define $\mathcal{M}_K = \{i : |y_i^*(y_{i|E_i})| \leq K |y_{i|E_i}|_2 / |y_i|_2\}$. Observe that because $||y_i^*|| \leq ||T^{-1}|| \, ||Q||$,

$$
\begin{aligned}
|y_i^*(y_{i|F_i \setminus E_i})| &\leq ||T^{-1}|| \, ||Q|| \max\{|y_{i|F_i \setminus E_i}|_2, \ |y_{i|F_i \setminus E_i}|_p\} \\
&\leq ||T^{-1}|| \, ||Q|| \max\{|y_i|_2, \rho^{1-2/p}\}.
\end{aligned}
$$

Thus if we consider only those y_i with $|y_i|_2 \leq \rho^{1-2/p}$ we have that

$$
|y_i^*(y_{i|E_i})| \geq 1 - ||T^{-1}|| \, ||Q|| \rho^{1-2/p}.
$$

Under our assumption on the sequence (y_i), the span of such y_i is still isomorphic to $X_{p,w}$. From now on we will assume that $\rho^{1-2/p} \leq (||T^{-1}|| \, ||Q|| 2)^{-1}$ and thus that $|y_i^*(y_{i|E_i})| \geq 1/2$ for all i. In this way we can work with $\mathcal{M}_K$ instead of $\mathcal{E}_\delta$ since for such i if $i \in \mathcal{M}_K$ then $i \in \mathcal{E}_{1/2K}$.

Let us now see how the projection onto $[y_i]$ acts on the span of $[y_{i|E_i}]$. Our assumptions on the y_i's imply that $Q \sum_{i=1}^{\infty} a_i y_{i|E_i} = \sum_{i=1}^{\infty} a_i y_i^*(y_{i|E_i}) y_i$. Hence

$$
\begin{aligned}
\left| Q \sum_{i=1}^{\infty} a_i y_{i|E_i} \right|_2 &= \left| \sum_{i=1}^{\infty} a_i y_i^*(y_{i|E_i}) y_i \right|_2 \\
&\geq K \left[\sum_{i \notin \mathcal{M}_K} |a_i|^2 |y_{i|E_i}|_2^2 |y_i|_2^{-2} |y_i|_2^2 \right]^{1/2} = K \left[\sum_{i \notin \mathcal{M}_K} |a_i|^2 |y_{i|E_i}|_2^2 \right]^{1/2}.
\end{aligned}
$$

If $a_i = |y_{i|E_i}|_2^{2/(p-2)}$, for $i \notin \mathcal{M}_K$ and $i \leq N$, and 0 else, then

$$\begin{aligned}
\Big|\Big|\sum_{i=1}^{N} a_i y_{i|E_i}\Big|\Big| &= \max\left\{\left[\sum_{i\notin\mathcal{M}_K} |a_i|^p |y_{i|E_i}|_p^p\right]^{1/p},\ \left[\sum_{i\notin\mathcal{M}_K} |a_i|^2 |y_{i|E_i}|_2^2\right]^{1/2}\right\}\\
&\le \max\left\{\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} |y_i|_p^p\right]^{1/p},\ \left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2}\right\}\\
&\le \max\left\{\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/p},\ \left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2}\right\}\\
&= \sum_{i\notin\mathcal{M}_K}\left[|y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2}, \qquad \text{if } \sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} \ge 1.
\end{aligned}$$

This implies that

$$||Q||\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2} \ge K\left[\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)}\right]^{1/2},$$

if $\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} \ge 1$. Therefore, if $K \ge ||Q||$,

$$\sum_{i\notin\mathcal{M}_K} |y_{i|E_i}|_2^{2p/(p-2)} \le 1.$$

Because $|y_{i|E_i}|_p \ge [1-\rho^{p-2}]^{1/p}$, (We are assuming that $\rho < 1$.) this implies that $[y_{i|E_i}||y_{i|E_i}||^{-1} : i \notin \mathcal{M}_K]$ is equivalent to the basis of ℓ_p. Therefore for any ϵ' small enough only finitely many of the y_i's with $|y_i|_2 \ge \epsilon'$ have index not in $\mathcal{M}_K$. Indeed, if this were not the case, then there would be a subsequence of (y_i), say $(y_i)_{i\in M}$, such that $M \subset \mathcal{M}_K^c$ and $\epsilon > 0, |y_i|_2 \ge \epsilon$ for all $i \in M$. However this would imply that Q is an isomorphism from $[y_{i|E_i} : i \in M]$, which is isomorphic to ℓ_p, onto $[y_i : i \in M]$, which is isomorphic to ℓ_2.

It follows that $(y_i)_{i\in\mathcal{M}_K}$ is equivalent to the basis of $X_{p,w'}$, where $w' = (w'_i)$ and for each $\epsilon > 0$ there is a $\epsilon' > 0$ such that $\epsilon > w'_i \ge \epsilon'$, for infinitely many i. Note that because $|y_i^*(y_{i|E_i})| \ge 1/2$, for any $i \in \mathcal{M}_K$, $|y_{i|E_i}|_2 \ge \delta|y_i|_2$, where $\delta = 1/2K$, i.e., $i \in \mathcal{E}_\delta$.

□

Remark 21. *If X is not isomorphic to X_p we can use part of the proof that 1) implies 2) to get a natural way of splitting vectors in X into a piece with large ratio and a piece with small ratio. Indeed suppose X is a complemented subspace of X_p and that P is the projection onto X. By [99] we may assume that P is bounded in the norm $|\cdot|_2$ as well. Suppose that 2) fails for $\delta < |P|_2^{-1}$, c and ϵ. Choose positive constants ϵ', ρ, α, and β such that*

$$\begin{aligned}
\epsilon' &< \min\{\epsilon, \delta\alpha\},\\
\beta &< \min\{(1-\delta|P|_2)/(||P||),\ \epsilon/c\},\\
\rho &\le \min\{c^{-p/(p-2)}\delta^{2/(p-2)},\ \beta^{p/(p-2)}\},\\
\beta > \alpha &\ge \max\{\beta\delta|P|_2/(1-\beta||P||),\ \beta^2||P||/(1-\delta|P|_2).
\end{aligned}$$

Let N be an integer so that a), b), and c) of 2) fail for ϵ' and N and suppose that $x \in X$, $x|_{[1,N]} = 0, \alpha < r(x) < \beta$ and $||x|| = 1$. Let $E_x = \{j : |x(j)| \ge \rho w_j^{2/(p-2)}|x|_2^{-2/(p-2)}\}$. As in the proof above the choice of ρ guarantees that the middle inequality in 2) c) is satisfied by $x_{|E_x}$. Because $r(x) < \beta \leq \epsilon/c$, the first inequality in 2) c) is also satisfied. Finally if $|x_{|E_x}|_2 \geq \delta |x|_2$, $\omega(E_x)^{(p-2)/2p} \geq |x_{|E_x}|_2 \geq \delta |x|_2 \geq \delta\alpha \geq \epsilon'$ and thus all of the inequalities in c) are satisfied. The failure of 2) then implies that $|x_{|E_x}|_2 < \delta |x|_2$.

Let $y = P(x_{|E_x})$ and $z = x - y = P(x_{|E_x^c})$. We claim that $r(y) \leq \alpha$ and $r(z) \geq \beta$. Indeed,

$$\begin{aligned} |y|_2 &\leq |P|_2 |x_{|E_x}|_2 < |P|_2 \delta |x|_2 \leq |P|_2 \delta \beta \\ &\leq \alpha(1 - \beta ||P||) \leq \alpha(|x|_p - ||P|| \, ||x_{|E_x^c}||) \leq \alpha |y|_p \end{aligned}$$

since $|x_{|E_x^c}|_p \leq \rho^{(p-2)/p} \leq \beta$ and $|x|_2 \leq \beta$, and

$$|z|_2 \geq (1 - \delta|P|_2)\alpha \geq (1 - \delta|P|_2)\beta^2 ||P||/(1 - \delta|P|_2) \geq \beta ||P|| \, ||x_{|E_x^c}|| \geq \beta |z|_p.$$

Thus any $x \in X$ with support in $\{N+1, N+2, \dots\}$ can be split into an element with ratio greater than β and one with ratio smaller than α. If this could be accomplished in a linear fashion it would follow that X is then isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$.

3.3.3 Complemented subspaces of X_p which are isomorphic to $\ell_p \oplus \ell_2$

In this section we look at some ways of discriminating between complemented subspaces of X_p which are isomorphic to complemented subspaces of $\ell_p \oplus \ell_2$ and those isomorphic to X_p. First we will examine how the conditions in Theorem 1.3 fail if X is isomorphic to $\ell_p \oplus \ell_2$. Below P_n denotes the basis projection onto the span of the first n elements of the basis of X_p.

Proposition 3.3.3. *Suppose that Z, X, U, and W are subspaces of X_p such that $Z \subset X = U \oplus W$, U is isomorphic to ℓ_2 and W is isomorphic to ℓ_p. Suppose that Z has a normalized K unconditional basis (z_n). Let $\beta = \lim_{n\to\infty} r(z_n)$ and $\beta' = \lim_{n\to\infty} \inf\{b :$ for every $\epsilon > 0$ there exists $u \in U$ such that $||P_n u|| < \epsilon$ and $r(u) \leq b\}$. If $\beta > 0$, $\beta' \leq 1$, and P is a projection from X onto Z then $||P|| \geq \beta'/K\beta$.*

Proof. Let $z_n = u_n + w_n$ where $u_n \in U$ and $w_n \in W$. By passing to subsequences and a standard perturbation argument we may assume that (u_n) and (w_n) are block bases of the basis of X_p. (It could happen that $||w_n|| \to 0$, but then $\beta' \leq \beta$.) Moreover we may assume that the projection P composed with the corresponding basis projection Q acts disjointly with respect to the subsequence $(z_n)_{n\in M}$, i.e., QP is a projection onto $[z_n : n \in M]$ and $QPu_n = \tau_n z_n$ and $QPw_n = (1-\tau_n) z_n$. Because (w_n) is equivalent to the usual unit vector basis of ℓ_p, $p > 2$, $\beta > 0$, and (z_n) is equivalent to the usual unit vector basis of ℓ_2, it follows that $\tau_n \to 1$.

Because W is isomorphic to ℓ_p, $|w_n|_2 \to 0$ and thus $|u_n|_2 - |z_n|_2 \to 0$. Therefore

$$\begin{aligned} &\limsup ||u_n|| = \limsup \max\{|u_n|_2, \ |u_n|_p\} \leq \limsup \max\{\beta, \ |u_n|_2 / r(u_n)\} \\ &\leq \max\{\beta, \beta/\beta'\} = \beta/\beta'. \end{aligned}$$

Consequently $K||P||\beta/\beta' \geq \limsup ||Q|| \, ||P|| \, ||u_n|| \geq 1$. □ □

Corollary 3.3.4. *Suppose that X, U, and W are subspaces of X_p which satisfy the hypotheses of* ***Proposition 2.1*** *and X is complemented in X_p with projection P. Then for any c and δ and $\epsilon < \beta' c\delta / \max\{c, \delta^{-1}\}$, there is no ϵ', $0 < \epsilon' < \epsilon$, such that for every $N \in \mathbb{N}$ there is an $x \in X$, $||x|| = 1$ and a finite set $E \subset \{N, \ N+1, \ \dots\}$ such that*

1. $||x_{|[1,N]}|| < N^{-1}$

2. $|x_{|E}|_2 \geq \delta |x|_2$

3. $\epsilon \geq c|x_{|E}|_2 \geq \omega(E)^{(p-2)/2p} \geq \epsilon'$.

Proof. Suppose ϵ' exists for some c, δ, and ϵ. Then there is a sequence (z_n) of norm one vectors in X which is a perturbation of a block basis of the basis of X_p and disjoint sets (E_n) such that for all n

1. $|z_{n|E_n}|_2 \geq \delta |z_n|_2$;

2. $\epsilon \geq c|z_{n|E_n}|_2 \geq \omega(E_n)^{(p-2)/2p} \geq \epsilon'$.

Then $[z_n : n \in \mathbb{N}]$ is complemented in X_p by a projection of norm at most $\max\{c, \delta^{-1}\}$ and $r(z_n) \leq \epsilon/c\delta$. Thus by the previous proposition $||P|| \geq \beta' c\delta/\epsilon$ and hence

$$\epsilon \geq \beta' c\delta / \max\{c, \delta^{-1}\}.$$

□

We now turn our attention to the classification of the complemented subspaces of X_p. It was shown in [99] that if a complemented subspace of X_p has an unconditional basis then it is isomorphic to ℓ_p, ℓ_2, $\ell_p \oplus \ell_2$, or X_p. In [102] the same conclusion was established if X has a "$p, 2$ F.D.D." Thus it seems likely that same result holds without the additional assumptions. We will next look at some well known results but recast in terms of the ratio of the 2-norm and p-norm. To begin let us recall that results of Kadec and Pelczynski [103] give a natural criterion for isomorphs of ℓ_2 contained in $X_p, p > 2$, namely, a subspace X of X_p is isomorphic to ℓ_2 if and only if there is a constant $C > 0$ such that $r(x) = |x|_2/|x|_p \geq C$ for all $x \in X$, i.e., $h(X) \geq C$. It follows from [**?**] that if a complemented subspace of X_p does not contain ℓ_2 then it is isomorphic to ℓ_p. A standard gliding hump argument yields the following criterion. (Below Q_N denotes the projection onto the span of the basis vectors of X_p with index greater than N.)

Proposition 3.3.5. *A complemented subspace X of X_p is isomorphic to ℓ_p if and only if for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that if $x \in Q_N X$ then $r(x) < \epsilon$.*

Theorem 1.3 gives a criterion for identifying complemented subspaces isomorphic to X_p, however it seems to be rather difficult to formulate useful conditions which identify complemented subspaces isomorphic to $\ell_2 \oplus \ell_p$ or a complemented subspace of it. Here are some attempts at such criteria.

Proposition 3.3.6. *Let X be a complemented subspace of X_p and suppose that Z is a subspace of X and ϵ, β, and β' are positive constants with $\epsilon \leq 1$ such that*

1. for all $z \in Z$, $r(z) \geq \beta'$

2. if $x \in X$ and $r(x) > \beta$ then there exists $z \in Z$ such that $|x - z|_2 < \epsilon |x|_2$.

Then X is isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$ and conversely.

Proof. Condition:
a) implies that Z is isomorphic to ℓ_2. Let Y be the kernel of the orthogonal projection from X onto Z. If $y \in Y$ and $r(y) > \beta$ then by b) there exists a $z \in Z$ such that $|y - z|_2 < \epsilon |y|_2$. But y is orthogonal to z so we have that $|y|_2 \leq [|y|_2^2 + |z|_2^2]^{1/2} = |y - z|_2 < \epsilon |y|_2$, an impossibility. Therefore $r(y) \leq \beta$ for all $y \in Y$ and Y is isomorphic to ℓ_p.

If X is isomorphic to ℓ_2 or ℓ_p the converse follows easily from our earlier observations. Thus by the results of Edelstein and Wojtasczyk [104] we may assume that X is isomorphic to $\ell_p \oplus \ell_2$ and let U and W be the complementary subspaces with U isomorphic to ℓ_2 and W isomorphic to ℓ_p. Because U is isomorphic to ℓ_2 there is a constant $1 \geq \beta' > 0$ such that for all $u \in U,\ r(u) \geq \beta'$. We may also assume that W is the kernel of the orthogonal projection Q onto U.

Then for any $x \in X$,

$$\begin{aligned}
&|x - Qx|_2 \leq r(W)||x - Qx|| \leq r(W)(1 + ||Q||)||x|| \\
&\leq r(W)(1 + 1/\beta')|x|_2 \max\{1, 1/r(x)\}.
\end{aligned}$$

Because W is isomorphic to ℓ_p it has a basis, let R_n denote $I - Q$ composed with the basis projection onto the span of the first n elements of the basis of W. Let $K = \sup ||R_n||$. Choose n so large that if $Y = (I - R_n)W,\ r(Y) < (2(1 + 1/\beta')(1 + K))^{-1}$. Then if $x \in Y + Z$ the above computation shows that b) is satisfied with $\epsilon = 1/2$. Because R_n is finite rank there exists a $\beta > 1$ such that if $r(x) > \beta$ then $||R_n x|| < ||x||/4(1 + ||Q||)$. Now if $r(x) > \beta$,

$$\begin{aligned}
&|x - Qx|_2 \leq (1 + ||Q||)||R_n x|| + |(I - R_n)x - Q(I - R_n)x|_2 \\
&\leq (1 + ||Q||)||x||/4(1 + 1/\beta') + r(Y)||(I - R_n)x - Q(I - R_n)x|| \\
&\leq |x|_2 \max\{1, 1/r(x)\}/4 + r(Y)(1 + ||Q||)||I - R_n||\ ||x|| \\
&\leq |x|_2 \max\{1, 1/r(x)\}/4 + r(Y)(1 + 1/\beta')(1 + K)|x|_2 \max\{1, 1/r(x)\}. \\
&\leq (3/4)|x|_2. \quad \square
\end{aligned}$$

□

Remark 22. *Condition b) may be replaced by*

1- if $x \in X$ and $r(x) > \beta''$ then there exists $z \in Z$ such that $|x - z|_2 < \epsilon ||x||$.

To see this note that if b') holds then b) holds with $\beta = \max\{\beta'', \epsilon\}$ and $\epsilon = 1$.

To get a similar theorem but with the hypothesis on the ℓ_p part we seem to need to assume the existence of a projection.

Proposition 3.3.7. *Let X be a complemented subspace of X_p and suppose that Y is the range of a projection P on X and ϵ, α, and α' are positive constants with $\epsilon \leq ||I - P||^{-1}$ such that*

1. *for all $y \in Y,\ r(y) \leq \alpha'$*
2. *if $x \in X$ and $r(x) < \alpha$ then there exists $y \in Y$ such that $||x - y|| < \epsilon ||x||$.*

Then X is isomorphic to a complemented subspace of $\ell_p \oplus \ell_2$ and conversely.

Proof. Condition a) implies that Y is isomorphic to ℓ_p. Let Z be the kernel of the projection P from X onto Y. If $z \in Z$ and $r(z) < \alpha$ then by b) there exists a $y \in Y$ such that $||z - y|| < \epsilon ||z||$.

But $Pz = 0$ and $Py = y$ so we have that $||z|| = ||(I-P)(z-y)|| < ||I-P||\epsilon||z|| \leq ||z||$, an impossibility. Therefore $r(z) \geq \alpha$ for all $z \in Z$ and Z is isomorphic to ℓ_2.

As in the proof of Proposition 2.6 the converse easily reduces to the case that X is isomorphic to $\ell_2 \oplus \ell_p$. So we again let U and W be the complementary subspaces with U isomorphic to ℓ_2 and W isomorphic to ℓ_p and let β' be a constant such that $1 \geq \beta' > 0$ and for all $u \in U, r(u) \geq \beta'$. As before we will assume that W is the kernel of the orthogonal projection Q onto U.

Then for any $x \in X$,

$$\begin{aligned} ||Qx|| &= \max\{|Qx|_2, |Qx|_p\} \leq \max\{|Qx|_2, |Qx|_2/\beta'\} \leq < Qx, x >^{1/2} /\beta' \\ &\leq [|Qx|_2 r(x)||x||]^{1/2}/\beta' \leq ||Q||^{1/2} r(x)^{1/2}||x||/\beta' \end{aligned}$$

Thus if $r(x) < \alpha = \beta'^2/||Q||^3$, $||x - (I-Q)x|| = ||Qx|| < ||x||/||I-(I-Q)||$. Because Y is isomorphic to ℓ_p there is some α' such that $r(y) \leq \alpha'$ for all $y \in Y$. ☐

Remark 23. *Propositions 2.3, 2.4, and 2.6 do not really use the structure of* X_p *and thus can be restated for complemented subspaces of* L_p*.*

Proposition 2.6 should be compared to the following result for X_p itself.

Proposition 3.3.8. *There does not exist a subspace* Y *of* X_p *and positive constants* ϵ *and* α, $\epsilon < 1$ *such that*

1. $r(Y) < \infty$
2. *if* $x \in X_p$ *and* $r(x) < \alpha$ *then there exists a* $y \in Y$ *with* $||x - y|| < \epsilon||x||$.

Proof. Suppose such a subspace exists. Then there is a normalized block basic sequence (x_n) of the X_p basis such that $\alpha > r(x_n) > \alpha/2$ for all n and such that $[x : n \in \mathbb{N}]$ is norm one complemented in X_p with projection P. By b) for each n there is an element y_n of Y such that $||x_n - y_n|| < \epsilon||x_n||$. Because P is norm 1, $||Py_n - x_n|| < \epsilon < 1$. Hence $||Py_n|| > 1 - \epsilon$. By passing to a subsequence we may assume that (y_n) is equivalent to the usual unit vector basis of ℓ_p and that (Py_n) is equivalent to a block basic sequence in $[x_n : n \in \mathbb{N}]$. But (x_n) is equivalent to the unit vector basis of ℓ_2 and hence so is (Py_n). Because $p > 2$ this is a contradiction. ☐

Remark 24. *The above proposition fails if* $\epsilon = 1$*. In this case the span of a perturbation of a natural basic sequence equivalent to the basis for* ℓ_p *may be used for* Y*.*

3.4 p-Summing Operators on Injective Tensor Products of Spaces

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$^{(*)}$ Research supported in part by an NSF Grant DMS 9001796.

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Abstract:
Let X, Y and Z be Banach spaces, and let $\prod_p(Y,Z)$ $(1 \leq p < \infty)$ denote the space of p-summing operators from Y to Z. We show that, if X is a $\mathcal{L}_\infty$-space, then a bounded linear operator $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if a naturally associated operator $T^\#: X \longrightarrow \prod_1(Y,Z)$ is 1-summing. This result need not be true if X is not a $\mathcal{L}_\infty$-space. For $p > 1$, several examples are given with $X = C[0,1]$ to show that $T^\#$ can be p-summing without T being p-summing. Indeed, there is an operator T on $C[0,1]\hat{\otimes}_\epsilon \ell_1$ whose associated operator $T^\#$ is 2-summing, but for all $N \in \mathbb{N}$, there exists an N-dimensional subspace U of $C[0,1]\hat{\otimes}_\epsilon \ell_1$ such that T restricted to U is equivalent to the identity operator on ℓ_∞^N.

Finally, we show that there is a compact Hausdorff space K and a bounded linear operator $T: C(K)\hat{\otimes}_\epsilon \ell_1 \longrightarrow \ell_2$ for which $T^\#: C(K) \longrightarrow \prod_1(\ell_1,\ell_2)$ is not 2-summing.

A.M.S. (1980) subject classification: 46B99.

3.4.1 Introduction.

Let X and Y be Banach spaces, and let $X\hat{\otimes}_\epsilon Y$ denote their injective tensor product. In this paper, we shall study the behavior of those operators on $X\hat{\otimes}_\epsilon Y$ that are p-summing.

If X, Y and Z are Banach spaces, then every p-summing operator $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ induces a p-summing linear operator $T^\#: X \longrightarrow \prod_p(Y,Z)$. This raises the following question: given two Banach spaces Y and Z, and $1 \leq p < \infty$, for what Banach spaces X is it true that a bounded linear operator $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is p-summing whenever $T^\#: X \longrightarrow \prod_p(Y,Z)$ is p-summing?

In [105], it was shown that whenever $X = C(\Omega)$ is a space of all continuous functions on a compact Hausdorff space Ω, then $T: C(\Omega)\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if $T^\#: C(\Omega) \longrightarrow \prod_1(Y,Z)$ is 1-summing. We will extend this result by showing that this result still remains true if X is any $\mathcal{L}_\infty$-space. We will also give an example to show that the result need not be true if X is not a $\mathcal{L}_\infty$-space. For this, we shall exhibit a 2-summing operator T on $\ell_2\hat{\otimes}_\epsilon \ell_2$ that is not 1-summing, but such that the associated operator $T^\#$ is 1-summing.

The case $p > 1$ turns out to be quite different. Here, the $\mathcal{L}_\infty$-spaces do not seem to play any important role. We show that for each $1 < p < \infty$, there exists a bounded linear operator $T: C[0,1]\hat{\otimes}_\epsilon \ell_2 \longrightarrow \ell_2$ such that $T^\#: C[0,1] \longrightarrow \prod_p(\ell_2,\ell_2)$ is p-summing, but such that T is not p-summing. We will also give an example that shows that, in general, the condition on $T^\#$ to be 2-summing is too weak to imply any good properties for the operator T at all. To illustrate this, we shall exhibit a bounded linear operator T on $C[0,1]\hat{\otimes}_\epsilon \ell_1$ with values in a certain Banach space Z, such that $T^\#: C[0,1] \longrightarrow \prod_2(\ell_1,Z)$ is 2-summing, but for any given $N \in \mathbb{N}$, there

exists a subspace U of $C[0,1]\hat{\otimes}_\epsilon \ell_1$, with $\dim U = N$, such that T restricted to U is equivalent to the identity operator on ℓ_∞^N.
Finally, we show that there is a compact Hausdorff space K and a bounded linear operator $T:\ C(K)\hat{\otimes}_\epsilon \ell_1 \longrightarrow \ell_2$ for which $T^\#:\ C(K) \longrightarrow \prod_1(\ell_1, \ell_2)$ is not 2-summing.

3.4.2 Definitions and Preliminaries

Let E and F be Banach spaces, and let $1 \leq q \leq p < \infty$. An operator $T:\ E \longrightarrow F$ is said to be (p,q)-**summing** if there exists a constant $C \geq 0$ such that for any finite sequence $e_1, e_2, \ldots, e_n$ in E, we have

$$\left(\sum_{i=1}^{n} \| T(e_i) \|^p\right)^{\frac{1}{p}} \leq C \sup\left\{\left(\sum_{i=1}^{n} |e^*(e_i)|^q\right)^{\frac{1}{q}} : e^* \in E^*, \| e^* \| \leq 1\right\}.$$

We let $\pi_{p,q}(T)$ denote the smallest constant C such that the above inequality holds, and let $\prod_{p,q}(E,F)$ be the space of all (p,q)-summing operators from E to F with the norm $\pi_{p,q}$. It is easy to check that $\prod_{p,q}(E,F)$ is a Banach space. In the case $p = q$, we will simply write $\prod_p(E,F)$ and π_p. We will use the fact that $T \in \prod_{p,q}(E,F)$ if and only if $\sum_n \|Te_n\|^p < \infty$ for every infinite sequence (e_n) in E with $\sum_n |e^*(e_n)|^q < \infty$ for each $e^* \in E^*$. That is to say, T is in $\prod_{p,q}(E,F)$ if and only if T sends all weakly ℓ_q-summable sequences into strongly ℓ_p-summable sequences. In what follows we shall mainly be interested in the case where $p = q$ and $p = 1$ or 2.

Given two Banach spaces E and F, we will let $E\hat{\otimes}_\epsilon F$ denote their injective tensor product, that is, the completion of the algebraic tensor product $E \otimes F$ under the cross norm $\|\cdot\|_\epsilon$ given by the following formula. If $\sum_{i=1}^{n} e_i \otimes x_i \in E \otimes F$, then

$$\| \sum_{i=1}^{n} e_i \otimes x_i \|_\epsilon = \sup\left\{\left|\sum_{i=1}^{n} e^*(e_i) x^*(x_i)\right| : \| e^* \| \leq 1, \| x^* \| \leq 1, e^* \in E^*, x^* \in F^*\right\}.$$

We will say that a bounded linear operator T between two Banach spaces E and F is called an **integral operator** if the bilinear form τ defines an element of $(E\hat{\otimes}_\epsilon F^*)^*$, where τ is induced by T according to the formula $\tau(e, x^*) = x^*(Te)$ ($e \in E$, $x^* \in F^*$). We will define the **integral norm** of T, denoted by $\| T \|_{\mathrm{int}}$, by

$$\| T \|_{\mathrm{int}} = \sup\left\{\left|\sum_{i=1}^{n} x_i^*(Te_i)\right| : \| \sum_{i=1}^{n} e_i \otimes x_i^* \|_\epsilon \leq 1\right\}.$$

The space of all integral operators from a Banach space E into a Banach space F will be denoted by $I(E,F)$. We note that $I(E,F)$ is a Banach space under the integral norm $\| \quad \|_{\mathrm{int}}$.

We will say that a Banach space X is a $\$_\infty$-**space** if, for some $\lambda > 1$, we have that for every finite dimensional subspace B of X, there exists a finite dimensional subspace E of X containing B, and an invertible bounded linear operator $T:\ E \longrightarrow \ell_\infty^{\dim E}$ such that $\| T \| \, \| T^{-1} \| \leq \lambda$.

It is well known that for any Banach spaces E and F, if T is in $I(E,F)$, then it is also in $\prod_1(E,F)$, with $\pi_1(T) \leq \| T \|_{\mathrm{int}}$. But $I(E,F)$ is strictly included in $\prod_1(E,F)$. It was shown in [106, p.477] that a Banach space E is a $\$_\infty$-space if and only if for any Banach space F, we have that $I(E,F) = \prod_1(E,F)$. We will use this characterization of $\$_\infty$-spaces in the sequel.

Finally, we note the following characterization of 1-summing operators (called right semi-integral by Grothendieck in [5]), which will be used later.

Proposition 3.4.1. *Let E and F be Banach spaces. Then the following properties about a bounded linear operator T from E to F are equivalent:*

(i) T is 1-summing;

(ii) There exists a Banach space F_1, and an isometric injection $\varphi : \ F \longrightarrow F_1$, such that $\varphi \circ T : \ E \longrightarrow F_1$ is an integral operator.

For all other undefined notions we shall refer the reader to either [107–109].

3.4.3 1-Summing and Integral Operators.

Let X and Y be Banach spaces with injective tensor product $X\hat{\otimes}_\epsilon Y$. For a Banach space Z, any bounded linear operator $T : \ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ induces a linear operator $T^\#$ on X by:

$$T^\# x(y) = T(x \otimes y) \qquad (y \in Y).$$

It is clear that the range of $T^\#$ is the space $\$(Y, Z)$ of bounded linear operators from Y into Z, and that $T^\#$ is a bounded linear operator.

In this section, we are going to investigate the 1-summing operators, and the integral operators, on $X\hat{\otimes}_\epsilon Y$. We will use Proposition 1 to relate these two ideas together. First of all, we have the following result.

Theorem 3.4.2. *Let X, Y and Z be Banach spaces, and let $T : \ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ be a bounded linear operator. Denote by $i : \ Z \longrightarrow Z^{**}$ the isometric embedding of Z into Z^{**}. Then the following two properties are equivalent:*

(i) $T \in I(X\hat{\otimes}_\epsilon Y, Z)$;

(ii) $\hat{i} \circ T \in I(X, I(Y, Z^{**}))$, where $\hat{i} : \ I(Y, Z) \longrightarrow I(Y, Z^{**})$ is defined by $\hat{i}(U) = i \circ U$ for each $U \in I(Y, Z)$.

In particular, if $T^\# \in I(X, I(Y, Z))$, then $T \in I(X\hat{\otimes}_\epsilon Y, Z)$.

Proof. First, we show that $(X\hat{\otimes}_\epsilon Y)\hat{\otimes}_\epsilon Z^*$ and $X\hat{\otimes}_\epsilon(Y\hat{\otimes}_\epsilon Z^*)$ are isometrically isomorphic to one another. To see this, note that the algebraic tensor product is an associative operation, that is, $(X \otimes Y) \otimes Z^*$ and $X \otimes (Y \otimes Z^*)$ are algebraically isomorphic. Also, they are both generated by elements of the form $\sum\limits_{i=1}^{n} x_i \otimes y_i \otimes z_i^*$, where $x_i \in X$, $y_i \in Y$ and $z_i^* \in Z^*$. Now, if we let $B(X^*)$, $B(Y^*)$ and $B(Z^{**})$ denote the dual unit balls of X^*, Y^* and Z^{**} equipped with their respective weak* topologies, then the spaces $(X \otimes_\epsilon Y) \otimes_\epsilon Z^*$ and $X \otimes_\epsilon (Y \otimes_\epsilon Z^*)$ embed isometrically into $C\left(B(X^*) \times B(Y^*) \times B(Z^{**})\right)$ in a natural way, by

$$\langle \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^*, \quad (x^*, y^*, z^{**}) \rangle = \sum_{i=1}^{n} x^*(x_i) y^*(y_i) z^{**}(z_i^*),$$

where $\sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^*$ is in $(X \otimes_\epsilon Y) \otimes_\epsilon Z^*$ or $X \otimes_\epsilon (Y \otimes_\epsilon Z^*)$, and (x^*, y^*, z^{**}) is in the compact set $B(X^*) \times B(Y^*) \times B(Z^{**})$. Thus both spaces $(X \hat{\otimes}_\epsilon Y) \hat{\otimes}_\epsilon Z^*$ and $X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)$ can be thought of as the closure in $C\left(B(X^*) \times B(Y^*) \times B(Z^{**})\right)$ of the algebraic tensor product of X, Y and Z^*. Now let us assume that $T: X \hat{\otimes}_\epsilon Y \longrightarrow Z$ is an integral operator. Then the bilinear map τ on $X \hat{\otimes}_\epsilon Y \times Z^*$, given by $\tau(u, z^*) = z^*(Tu)$ for $u \in X \hat{\otimes}_\epsilon Y$ and $z^* \in Z^*$, defines an element of $\left(X \hat{\otimes}_\epsilon Y \hat{\otimes}_\epsilon Z^*\right)^*$, that is,

$$\parallel T \parallel_{It} = \sup \left\{ | \sum_{i=1}^{n} z_i^* \left(T(x_i \otimes y_i)\right) : \ \parallel \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \parallel_\epsilon \leq 1 \right\}.$$

To show that for every x in X the operator $T^\# x$ is in $I(Y, Z)$, with

$$\parallel T^\# x \parallel_{It} \leq \parallel x \parallel \ \parallel T \parallel_{It},$$

is easy. This is because, for each $x \in X$, the operator $T^\# x$ is the composition of T with the bounded linear operator from Y to $X \hat{\otimes}_\epsilon Y$, which to each y in Y gives the element $x \otimes y$. If $i : Z \longrightarrow Z^{**}$ denotes the isometric embedding of Z into Z^{**}, it induces a bounded linear operator $\hat{i} : I(Y, Z) \longrightarrow I(Y, Z^{**})$ given by $\hat{i}(U) = i \circ U$ for all $U \in I(Y, Z)$. It is immediate that $\hat{i}$ is an isometry. We will now show that the operator $\hat{i} \circ T^\# : X \longrightarrow I(Y, Z^{**})$ is integral. It is well known (see [109, p.237]) that the space $I(Y, Z^{**})$ is isometrically isomorphic to the dual space $(Y \hat{\otimes}_\epsilon Z^*)^*$. Thus to show that $\hat{i} \circ T^\# : X \longrightarrow (Y \hat{\otimes}_\epsilon Z^*)^*$ is an integral operator, we need to show that it induces an element of $\left(X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)\right)^*$. For this, it is enough to note that, by our discussion concerning the isometry of $(X \hat{\otimes}_\epsilon Y) \hat{\otimes}_\epsilon Z^*$ and $X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z^*)$, that

$$\parallel \hat{i} \circ T^\# \parallel_{It} = \sup \left\{ | \sum_{i=1}^{n} \hat{i} \circ T^\# x_i, y_i \otimes z_i^* | : \parallel \sum_{i=1}^{n} x_i \otimes y_i \otimes z_i^* \parallel_\epsilon \leq 1 \right\}.$$

But for each $x \in X$, $y \in Y$ and $z^* \in Z^*$, we have

$$\langle \hat{i} \circ T^\# x, y \otimes z^* \rangle = \langle T(x \otimes y), z^* \rangle.$$

Hence, from (*) and (**), it follows that

$$\parallel \hat{i} \circ T \parallel_{It} = \parallel T \parallel_{It} .$$

Thus we have shown that (i) $\Rightarrow$ (ii). The proof of (ii) $\Rightarrow$ (i) follows in a similar way. If $\hat{i} \circ T^\# : X \longrightarrow I(Y, Z^{**})$ is an integral operator, then one can show that $i \circ T : X \hat{\otimes}_\epsilon Y \longrightarrow Z^{**}$ is integral, which in turn implies that T itself is integral (see[109, p.233]). Finally, the last assertion follows easily, since if $T^\# : X \longrightarrow I(Y, Z)$ is integral, then $\hat{i} \circ T$ is integral (see [109, p.232]). □

Since the mapping $\hat{i} : I(Y, Z) \longrightarrow I(Y, Z^{**})$ is an isometry, Proposition 1 coupled with Theorem 2 implies that, if $T : X \hat{\otimes}_\epsilon Y \longrightarrow Z$ is an integral operator, then $T^\# : X \longrightarrow I(Y, Z)$ is 1-summing. This result can be shown directly from the definitions. In what follows we shall present a sketch of that alternative approach.

Theorem 3.4.3. *Let X, Y and Z be Banach spaces, and let $T : X \hat{\otimes}_\epsilon Y \longrightarrow Z$ be a bounded linear operator. If T is integral, then $T^\# : X \longrightarrow I(Y, Z)$ is 1-summing. If in addition X is a $\$_\infty$-space, then $T : X \hat{\otimes}_\epsilon Y \longrightarrow Z$ is integral if and only if $T^\# : X \longrightarrow I(Y, Z)$ is integral.*

Proof. First, we will show that, if $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is an integral operator, then $T^{\#}$ is in $\prod_1(X, I(Y,Z))$ with $\pi_1(T^{\#}) \leq \| T \|_{\text{int}}$. Let $x_1, x_2, \dots, x_n$ be in X, and fix $\epsilon > 0$. For each $i \leq n$, there exists $n_i \in \mathbb{N}$, $(y_{ij})_{j\leq n_i}$ in Y, and $(z^*_{ij})_{j\leq n_i}$ in Z^*, such that $\| \sum\limits_{j=1}^{ni} y_{ij} \otimes z^*_{ij} \|_\epsilon \leq 1$, and

$$\| T^{\#} x_i \|_{\text{int}} \leq \sum_{j=1}^{n_i} z^*_{ij}\left(T(x_i \otimes y_{ij})\right) + \frac{\epsilon}{2^i}.$$

Since T is an integral operator, and

$$\| \sum_{i=1}^{n} \sum_{j=1}^{n_i} x_i \otimes y_{ij} \otimes z^*_{ij} \|_\epsilon \leq \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \| x^* \| \leq 1, x^* \in X^* \right\},$$

it follows that

$$\sum_{i=1}^{n} \sum_{j=1}^{n_i} z^*_{ij}\left(T(x_i \otimes y_{ij})\right) \leq \| T \|_{\text{int}} \ \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \| x^* \| \leq 1, \ x^* \in X^* \right\}.$$

Therefore

$$\sum_{i=1}^{n} \| T^{\#} x_i \|_{\text{int}} \leq \| T \|_{\text{int}} \ \sup \left\{ \sum_{i=1}^{n} |x^*(x_i)| : \ x^* \in X^*, \| x^* \| \leq 1 \right\} + \epsilon.$$

Now, if in addition X is a $\mathcal{\$}_\infty$-space, then by [106, p.477], the operator $T^{\#}$ is indeed integral. □

Remark 25. *f $X = C(\Omega)$ is a space of continuous functions defined on a compact Hausdorff space Ω, one can deduce a similar result to Theorem 3 from the main result of [110].*

Our next result extends a result of [111] to $\mathcal{\$}_\infty$-spaces, where it was shown that whenever $X = C(\Omega)$, a space of all continuous functions on a compact Hausdorff space Ω, then a bounded linear operator $T: C(\Omega)\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if $T^{\#}: C(\Omega) \longrightarrow \prod_1(Y,Z)$ is 1-summing. This also extends a result of [112] where similar conclusions were shown to be true for $X = A(K)$, a space of continuous affine functions on a Choquet simplex K (see [**?**]).

We note that one implication follows with no restriction on X. If X, Y and Z are Banach spaces, and $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is a 1-summing operator, then $T^{\#}$ takes its values in $\prod_1(Y,Z)$. This follows from the fact that for each $x \in X$, the operator $T^{\#}x$ is the composition of T with the bounded linear operator from Y into $X\hat{\otimes}_\epsilon Y$ which to each y in Y gives the element $x \otimes y$ in $X\hat{\otimes}_\epsilon Y$, and hence

$$\pi_1(T^{\#}x) \leq \| x \| \ \pi_1(T).$$

Moreover, one can proceed as in [111] to show that $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is 1-summing.

Theorem 3.4.4. *If X is a $\mathcal{\$}_\infty$ space, then for any Banach spaces Y and Z, a bounded linear operator $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 1-summing if and only if $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is 1-summing.*

Proof. Let $T: X\hat{\otimes}_\epsilon Y \longrightarrow Z$ be such that $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is 1-summing. Since X is a $\mathcal{\$}_\infty$-space, it follows from [112, p.477] that $T^{\#}: X \longrightarrow \prod_1(Y,Z)$ is an integral operator. Let φ

denote the isometric embedding of Z into $C\left(B(Z^*)\right)$, the space of all continuous scaler functions on the unit ball $B(Z^*)$ of Z^* with its weak*-topology. This induces an isometry

$$\hat{\varphi}: \prod\nolimits_1(Y,Z) \longrightarrow \prod\nolimits_1\left((Y, C(B(Z^*))\right),$$

$$\hat{\varphi}(U) = \varphi \circ U \qquad \text{for all } U \in \prod\nolimits_1(Y,Z).$$

Now, it follows from[**?** , p.301], that $\prod_1\left(Y, C(B(Z^*))\right)$ is isometric to $I\left(Y, C(B(Z^*))\right)$. Hence we may assume that $\hat{\varphi}\circ T^{\#}:\ X \longrightarrow I\left(Y, C(B(Z^*))\right)$ is an integral operator. Moreover, it is easy to check that $(\varphi\circ T)^{\#} = \hat{\varphi}\circ T^{\#}$. By Theorem 2 the operator $\varphi\circ T:\ X\hat{\otimes}_\epsilon Y \longrightarrow C(B(Z^*))$ is an integral operator, and hence T is in $\prod_1\left(X\hat{\otimes}_\epsilon Y, Z\right)$ by Proposition 1. ☐

In the following section we shall, among other things, exhibit an example that illustrates that it is crucial for the space X to be a $\mathcal{L}_\infty$-space if the conclusion of Theorem 5 is to be valid.

3.4.4 2-summing Operators and some Counter-examples.

In this section we shall study the behavior of 2-summing operators on injective tensor product spaces. As we shall soon see, the behavior of such operators when $p = 2$ is quite different from when $p = 1$. For instance, unlike the case $p = 1$, the $\mathcal{L}_\infty$-spaces don't seem to play any particular role. In fact, we shall exhibit operators T on $C[0,1]\hat{\otimes}_\epsilon \ell_2$ which are not 2-summing, yet their corresponding operators $T^{\#}$ are. We will also give other interesting examples that answer some other natural questions.

We will present the next theorem for $p = 2$, but the same result is true for any $1 \le p < \infty$, with only minor changes.

Theorem 3.4.5. *Let X, Y and Z be Banach spaces. If $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is a 2-summing operator, then $T^{\#}:\ X \longrightarrow \prod_2(Y,Z)$ is a 2-summing operator.*

Proof. If $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is 2-summing, then using the same kind of arguments that we have given above, it can easily be shown that for each $x \in X$, that $T^{\#}x \in \prod_2(Y,Z)$, with $\pi_2(T^{\#}x) \le \pi_2(T) \parallel x \parallel$.

Now we will show that $T^{\#}:\ X \longrightarrow \prod_2(Y,Z)$ is 2-summing. Let (x_n) be in X such that $\sum\limits_n |x^*(x_n)|^2 < \infty$ for each x^* in X^*. Fix $\epsilon > 0$. For each $n \ge 1$, let (y_{nm}) be a sequence in Y such that

$$\sup\left\{\left(\sum_{m=1}^{\infty}|y^*(y_{nm})|^2\right)^{1/2} : \parallel y^* \parallel \le 1, y^* \in Y^*\right\} \le 1,$$

and

$$\pi_2\left(T^{\#}x_n\right) \le \left(\sum_{m=1}^{\infty}\parallel T(x_n\otimes y_{nm})\parallel^2\right)^{1/2} + \frac{\epsilon}{2^n}.$$

Then

$$\left[\pi_2\left(T^{\#}x_n\right)\right]^2 \le \sum_{m=1}^{\infty}\parallel T\left(x_n\otimes y_{nm}\right)\parallel^2 + \frac{\epsilon}{2^{n-1}}\left(\sum_{m=1}^{\infty}\parallel T\left(x_n\otimes y_{nm}\right)\parallel^2\right)^{1/2} + \frac{\epsilon^2}{2^{2n}}.$$

Now, consider the sequence $(x_n \otimes y_{nm})$ in $X\hat{\otimes}_\epsilon Y$. For each $\xi \in \left(X\hat{\otimes}_\epsilon Y\right)^* \simeq I(X, Y^*)$ we have that

$$\begin{aligned}\sum_{m,n} |\xi(x_n)(y_{nm})|^2 &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |\xi(x_n)(y_{nm})|^2 \\ &\leq \sum_{n=1}^{\infty} \parallel \xi(x_n) \parallel^2 .\end{aligned}$$

Since $\xi \in I(X, Y^*)$, it follows that $\xi \in \prod_2(X, Y^*)$, and so

$$\sum_{n=1}^{\infty} \parallel \xi(x_n) \parallel^2 < \infty.$$

Hence we have shown that for all $\xi \in (X\hat{\otimes}_\epsilon Y)^*$,

$$\sum_{m,n} |\xi(x_n)(y_{nm})|^2 < \infty.$$

Since $T \in \prod_2 \left(X\hat{\otimes}_\epsilon Y, Z\right)$, we have that

$$\sum_{m,n} \parallel T(x_n \otimes y_{nm}) \parallel^2 < \infty,$$

and therefore

$$\sum_{n} \left[\pi_2 \left(T^{\#} x_n\right)\right]^2 < \infty.$$

☐

Proof. The above result extends a result of [113], where it was shown that if $T:\ X\hat{\otimes}_\epsilon Y \longrightarrow Z$ is p-summing for $1 \leq p < \infty$, then $T^{\#}:\ X \longrightarrow \$(Y, Z)$ is p-summing. ☐

Now we shall give the example that we promised at the end of subsection 2.

Theorem 3.4.6. *There exists a bounded linear operator $T:\ \ell_2\hat{\otimes}_\epsilon \ell_2 \longrightarrow \ell_2$ such that T is not 1-summing, yet $T^{\#}:\ \ell_2 \longrightarrow \pi_1(\ell_2, \ell_2)$ is 1-summing.*

Proof. First, we note the well known fact that $\ell_2\hat{\otimes}_\epsilon \ell_2 = slK(\ell_2, \ell_2)$, the space of all compact operators from ℓ_2 to ℓ_2. Now we define T as the composition of two operators. Let $P:\ slK(\ell_2, \ell_2) \longrightarrow c_0$ be the operator defined so that for each $K \in slK(\ell_2, \ell_2)$,

$$P(K) = (K(e_n)(e_n)),$$

where (e_n) is the standard basis of ℓ_2. It is well known [108, p.145] that the sequence $(e_n \otimes e_n)$ in $\ell_2\hat{\otimes}_\epsilon \ell_2$ is equivalent to the c_0-basis, and that the operator P defines a bounded linear projection of $slK(\ell_2, \ell_2)$ onto c_0.

Let $S:\ c_0 \longrightarrow \ell_2$ be the bounded linear operator such that for each $(\alpha_n) \in c_0$

$$S(\alpha_n) = \left(\frac{\alpha_n}{n}\right).$$

It is easily checked [107, p.39] that S is a 2-summing operator that is not 1-summing.

Now we define $T: \ slK(\ell_2, \ell_2) \longrightarrow \ell_2$ to be $T = S \circ P$. Thus T is 2-summing but not 1-summing. It follows from Theorem 6 that the induced operator $T^{\#}: \ell_2 \longrightarrow \prod_2(\ell_2, \ell_2)$ is 2-summing. Since ℓ_2 is of cotype 2, it follows from [108, p.62], that for any Banach space E, we have $\prod_2(\ell_2, E) = \prod_1(\ell_2, E)$, and that there exists a constant $C > 0$ such that for all $U \in \prod_2(\ell_2, E)$ we have

$$\pi_1(U) \leq C\pi_2(U).$$

This implies that $T^{\#}$ is 1-summing as an operator taking its values in $\prod_1(\ell_2, \ell_2)$. $\square$

Remark 26. *We do not need to use Theorem 6 to show that $T^{\#}$ is 1-summing in the example above. Instead, we can use the following argument. First note that $T^{\#}$ factors as follows:*

$$\begin{tikzcd}
\ell_2 \arrow[r, "T^{\#}"] \arrow[d, "A"'] & \pi_1(\ell_2, \ell_2) \\
\ell_2 \arrow[ur, "B"'] &
\end{tikzcd}$$

Here $A: \ \ell_2 \to \ell_2$ is the 1-summing operator defined by

$$A(\alpha_n) = \left(\frac{\alpha_n}{n}\right),$$

for each $(\alpha_n) \in \ell_2$, and $B: \ \ell_2 \longrightarrow \pi_1(\ell_2, \ell_2)$ is the natural embedding of ℓ_2 into the space $\pi_1(\ell_2, \ell_2)$ defined by

$$B(\beta_n)(\gamma_n) = (\beta_n \gamma_n)$$

for each (β_n), $(\gamma_n) \in \ell_2$.

Now we will give two examples concerning the case when $p > 1$. We will show that we do not have a converse to Theorem 8, even when the underlying space X is a $\$_{\infty}$-space.

First, let us fix some notation. In what follows we shall denote the space $\ell_p(\mathbb{Z})$ by ℓ_p, and call its standard basis $\{e_n : n \in \mathbb{Z}\}$. Thus if $x = (x(n)) \in \ell_p$, then $x(n) = \langle x, e_n \rangle$, and

$$\| x \|_{\ell_p} = \left(\sum_{n=1}^{\infty} |\langle x, e_n \rangle|^p \rangle \right)^{\frac{1}{p}}.$$

For $f \in L_p[0, 1]$, we let

$$\| f \|_{L_p} = \left(\int_0^1 |f(t)|^p dt \right)^{\frac{1}{p}}.$$

If Ω is a compact Hausdorff space, and Y is a Banach space, then $C(\Omega, Y) = C(\Omega) \hat{\otimes}_{\epsilon} Y$ will denote the Banach space of continuous Y-valued functions on Ω under the supremum norm.

We recall that since ℓ_2 is of cotype 2, we have that $\prod_2(\ell_2, \ell_2) = \prod_1(\ell_2, \ell_2)$. We also recall that, if $u = \sum\limits_{n=1}^{\infty} \alpha_n e_n \otimes e_n$ is a diagonal operator in $\prod_2(\ell_2, \ell_2)$, then

$$\pi_2(u) = \left(\sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{\frac{1}{2}} = \text{ the Hilbert-Schmidt norm of } u.$$

Theorem 3.4.7. *For each $1 < p < \infty$, there is a bounded linear operator $T : \ C([0,1], \ell_2) \to \ell_2$ that is not p-summing, but such that $T^{\#} : \ C[0,1] \longrightarrow \Pi_p(\ell_2, \ell_2)$ is p-summing.*

Proof. We present the proof for $p \leq 2$. The case where $p > 2$ follows by the same argument. For each $n \in \mathbb{Z}$, let $\epsilon_n(t) : \ [0,1] \to \mathbb{C}, \ \epsilon_n(t) = e^{2\pi It}$ denote the standard trigonometric basis of $L_2[0,1]$. If $f \in L_1[0,1]$, let $\hat{f}(n) = \int_0^1 f(t)\epsilon_n(t)dt$ denote the usual Fourier coefficient of f. For each $\lambda = (\lambda_n)$, where $|\lambda_n| \leq 1$ for all $n \in \mathbb{Z}$, define the operator

$$T_\lambda : \ C\left([0,1], \ell_2\right) \longrightarrow \ell_2$$

such that for $\varphi \in C\left([0,1], \ell_2\right)$ we have

$$T_\lambda \varphi = \left(\lambda_n \ \langle \hat{\varphi}(n), e_n \rangle \ \right).$$

Here $\hat{\varphi}(n) = \ \text{Bochner} - \int_0^1 \varphi(t)\epsilon_n(t)dt$. The operator T_λ is a bounded linear operator, with $\| T_\lambda \varphi \|_{\ell_2} \leq \| \varphi \|$. To see this, note that for $\varphi \in C\left([0,1], \ell_2\right)$ we have

$$\begin{aligned}
\| T_\lambda \varphi \|_{\ell_2}^2 &= \sum_n |\lambda_n|^2 | \langle \hat{\varphi}(n), e_n \rangle |^2 \\
&\leq \sum_n | \langle \hat{\varphi}(n), e_n \rangle |^2 \\
&\leq \sum_n \int_0^1 | \langle \varphi(t), e_n \rangle |^2 dt \\
&= \int_0^1 \| \varphi(t) \|_{\ell_2}^2 \, dt \\
&\leq \sup_t \| \varphi(t) \|_{\ell_2}^2 .
\end{aligned}$$

Now, note that if $f \in C\left([0,1]\right)$, and $x \in \ell_2$, then

$$T_\lambda(f \otimes x) = \Big(\lambda_n \hat{f}(n) \langle x, e_n \rangle\Big),$$

and hence the operator $T_\lambda^{\#} : \ C[0,1] \to \$(\ell_2, \ell_2)$ is such that

$$T_\lambda^{\#} f(x) = \Big(\lambda_n \hat{f}(n) \langle x, e_n \rangle\Big).$$

Thus

$$\pi_2(T_\lambda^{\#} f) = \left(\sum_n |\lambda_n|^2 |\hat{f}(n)|^2\right)^{\frac{1}{2}}.$$

Hence, by Hölder's inequality,

$$\pi_2(T_\lambda^{\#} f) \leq \| (\lambda_n) \|_{\ell_r} \| (\hat{f}(n)) \|_{\ell_q},$$

where $\dfrac{1}{r} + \dfrac{1}{q} = \dfrac{1}{2}$. By the Hausdorff-Young inequality, we have that

$$\| (\hat{f}(n)) \|_{\ell_q} \leq \| f \|_{L_p},$$

where $1 \leq p \leq 2$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Thus

$$\pi_2(T_\lambda^{\#} f) \leq \| (\lambda_n) \|_{\ell_r} \ \| f \|_{L_p},$$

for $1 \leq p \leq 2, \ 2 \leq r \leq \infty$ and $\dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{2}$. This shows that if $\| (\lambda_n) \|_{\ell_r} < \infty$, then

(1) $T_\lambda^{\#}\,(C[0,1]) \subseteq \pi_2(\ell_2,\ell_2) = \pi_p(\ell_2,\ell_2)$;

(2) $T_\lambda^{\#} :\ C[0,1] \longrightarrow \pi_p(\ell_2,\ell_2)$ is p-summing.

Now, let $U \subset C\,([0,1],\ell_2)$ be the closed linear span of $\{\epsilon_i \otimes e_i,\ a_i \in \mathbb{Z}\}$. Then U is isometrically isomorphic to ℓ_2. This is because

$$\begin{aligned} \parallel \sum_i \mu_i \epsilon_i \otimes e_i \parallel &= \sup_{t\in[0,1]} \parallel (\mu_n \epsilon_n(t)) \parallel_{\ell_2} \\ &= \parallel (\mu_i \epsilon_i(t_0)) \parallel_{\ell_2}, \end{aligned}$$

for some $t_0 \in [0,1]$, and hence

$$\parallel \sum_i \mu_i \epsilon_i \otimes e_i \parallel = \left(\sum_i |\mu_i|^2\right)^{\frac{1}{2}}.$$

Moreover

$$T_\lambda(\epsilon_i \otimes e_i) = \lambda_i e_i \qquad \text{for all } i \in \mathbb{Z},$$

Therefore, we have the following commuting diagram

$$\begin{tikzcd} U \arrow[r, "T_{\lambda|U}"] \arrow[d, "Q"'] & \ell_2 \\ \ell_2 \arrow[ur, "S_\lambda"'] & \end{tikzcd}$$

where $Q :\ U \to \ell_2$ is the isomorphism from U onto ℓ_2 such that $Q(\epsilon_n \otimes e_n) = e_n$ for all $n \in \mathbb{Z}$, and $S_\lambda :\ \ell_2 \longrightarrow \ell_2$ is the operator given by $S_\lambda(e_n) = \lambda_n e_n$. So to show that T_λ is not p-summing, it is sufficient to show that one can pick $\lambda = (\lambda_n)$ such that S_λ is not p-summing. To do this, we consider two cases. If $p = 2$, we take $\lambda_n = 1$ for all $n \in \mathbb{Z}$. Then the map S_λ induced on ℓ_2 is the identity map which is not s-summing for any $s < \infty$. If $1 < p < 2$, let $\lambda_n = \dfrac{1}{|n+1|^{\frac{1}{r}} \log|n+1|}$, so that $\parallel (\lambda_n) \parallel_{\ell_r} < \infty$. Then the map $S_\lambda :\ \ell_2 \longrightarrow \ell_2$ is not s-summing for any $s < r$. To show this, we may assume, without loss of generality, that $s \geq 2$. Let $x_n = e_n$ for all $n \geq 1$, and note that

$$\sup_{x^*\in B(\ell_2)} \left(\sum_n |x^*(x_n)|^s\right)^{\frac{1}{s}} \leq \parallel x^* \parallel_{\ell_2} \leq 1,$$

whilst

$$\left(\sum_n \parallel \lambda_n x_n \parallel^s\right)^{\frac{1}{s}} = \infty.$$

□

While the operators T_λ in the previous example failed to be p-summing, they were all (2,1)-summing. This suggests the following question: suppose $T :\ C\,([0,1],Y) \longrightarrow Z$ is a bounded linear operator such that $T^{\#} :\ C[0,1] \longrightarrow \prod_2(Y,Z)$ is 2-summing. What can we say about T? Is T $(2,1)$-summing? The following example shows that T can be very bad.

Theorem 3.4.8. *There exists a Banach space Z, and a bounded linear operator $T:\ C([0,1],\ell_1)\to Z$ such that $T^{\#}:\ C[0,1]\to\prod_2(\ell_1,Z)$ is 2-summing, with the property that, for any $N\in\mathbb{N}$, there exists a subspace U of $C([0,1],\ell_1)$ with $\dim U=N$, such that T restricted to U behaves like the identity operator on ℓ_∞^N. In particular T is not (2,1)-summing.*

Proof. If X and Y are Banach spaces, we denote by $X\hat{\otimes}_\pi Y$ the projective tensor product, that is, the completion of the algebraic tensor product of X and Y under the norm

$$\| u \|_\pi = \inf\{\sum_{i=1}^{n} \| x_i \| \| y_i \|,\ u=\sum_{i=1}^{n} x_i\otimes y_i\}.$$

It is well known that $(X\hat{\otimes}_\pi Y)^*$ is isometrically isomorphic to the space $\$(X,Y^*)$ of all bounded linear operators from X to Y^*.

Let $Z=C([0,1],\ell_1)+L_2[0,1]\hat{\otimes}_\pi\ell_2$ be the Banach space with the norm

$$\| x \|_Z = \inf\{\| x' \|_\epsilon + \| x'' \|_\pi :\ x=x'+x''\},$$

where $\| \ \|_\epsilon$ denotes the sup norm in $C([0,1],\ell_1)$, and $\| \ \|_\pi$ denotes the norm of the projective tensor product $L_2[0,1]\hat{\otimes}_\pi\ell_2$. Let

$$T:\ C([0,1],\ell_1)\longrightarrow Z$$

be the identity operator.

We first see that for each $f\in C[0,1]$, the operator $T^{\#}f:\ \ell_1\to Z$ is 2-summing with

$$\pi_2(T^{\#}f)\le\pi_2(I)\ \| T^{\#}f \|_{\$(\ell_2,Z)},$$

where $I:\ell_1\longrightarrow\ell_2$ is the natural mapping. This is because, for each $f\in C[0,1]$, and each $x\in\ell_1$, we have that

$$\| T(f\otimes x) \| \le \| f\otimes x \|_{L_2\hat{\otimes}_\pi\ell_2} \le \| f \|_{L_2}\ \| x \|_{\ell_2}.$$

To see that $T^{\#}:\ C[0,1]\longrightarrow\prod_2(\ell_1,X)$ is 2-summing, note that $\| T^{\#}f \|_{\$(\ell_2,Z)}\le\| f \|_{L_2}$, and hence if $f_1,\dots,f_n\in C[0,1]$, then

$$\begin{aligned}\left(\sum_{k=1}^{n}\left[\pi_2(T^{\#}f_k)\right]^2\right)^{\frac{1}{2}} &\le \pi_2(I)\left(\sum_{k=1}^{n}\| f_k \|_{L_2}^2\right)^{\frac{1}{2}}\\ &\le \pi_2(I)\pi_2(J)\sup_{t\in[0,1]}\left\|\left(\sum_{K=1}^{n}|f_k(t)|^2\right)^{\frac{1}{2}}\right\|.\end{aligned}$$

Here $J:\ C[0,1]\longrightarrow L_2[0,1]$ denotes the natural mapping.

Now we define the space U, a closed linear subspace of $C([0,1],\ell_1)$. Let $\{f_{ij}:\ 1\le i,j\le N\}$ be disjoint functions in $C[0,1]$, for which $0\le f_{ij}\le 1$, $\| f_{ij} \|=1$, each f_{ij} is supported in an interval of length $\frac{1}{N^2}$, and

$$\int_0^1 f_{ij}dt=\frac{1}{2N^2}\ \text{and}\ \int_0^1 f_{ij}^2dt=\frac{1}{3N^2}.$$

Let $\{e_{ij}:\ 1\le i,j\le N\}$ be distinct unit vectors in ℓ_1. We let $U=\{\sum_{i,j}\lambda_i f_{ij}\otimes e_{ij},\ \lambda_i\in\mathbb{R}\}$.

Now we consider T restricted to U. If $\sum\limits_{i,j} \lambda_i f_{ij} \otimes e_{ij} \in U$, then

$$\| \sum_{i,j} \lambda_i f_{ij} \otimes e_{ij} \|_\epsilon \le \sup_i |\lambda_i|,$$

and hence

$$\| \sum_{i,j} \lambda_i f_{ij} \otimes e_{ij} \|_Z \le \sup_i |\lambda_i|.$$

Let $y_i^* = N \sum\limits_j f_{ij} \otimes e_{ij}$, and set $x = \sum\limits_{i,j} \lambda_i f_{ij} \otimes e_{ij}$. Then whenever $x = x' + x''$, with $x' \in C\left([0,1], \ell_1\right)$ and $x'' \in L_2[0,1]\hat{\otimes}_\pi \ell_2$, we know that

$$|y_i^*(x)| \le |y_i^*(x')| + |y_i^*(x'')|.$$

Hence

$$|y_i^*(x)| \le \| y_i^* \|_{C([0,1],\ell_1)^*} \| x' \|_\epsilon + \| y_i^* \|_{(L_2[0,1]\hat{\otimes}_\pi \ell_2)^*} \| x'' \|_\pi .$$

But

$$\begin{aligned} \| y_i^* \|_{C([0,1],\ell_1)^*} &= N \sum_{i=1}^N \int_{supp f_{ij}} |f_{ij}| dt \\ &= N \cdot \frac{N}{2N^2} = \frac{1}{2}, \end{aligned}$$

and, since $\left(L_2[0,1]\hat{\otimes}_\pi \ell_2\right)^*$ is isometric to $\$(L_2[0,1], \ell_2)$,

$$\begin{aligned} \| y_i^* \|_{(L_2[0,1]\hat{\otimes}_\pi \ell_2)^*} &= \sup \left\{ [\sum_{j=1}^N (N \int_0^1 f_{ij} g dt)^2]^{\frac{1}{2}} : \| g \|_{L_2} \le 1 \right\} \\ &\le \sup \left\{ N[\sum_{j=1}^N \int_0^1 f_{ij}^2 dt \cdot \int_{supp f_{ij}} |g|^2 dt]^{\frac{1}{2}} : \| g \|_{L_2} \le 1 \right\} \\ &= \frac{1}{\sqrt{3}} \left\{ (\sum_{j=1}^N \int_{supp f_{ij}} |g|^2 dt)^{\frac{1}{2}} : \| g \|_2 \le 1 \right\} \\ &= \frac{1}{\sqrt{3}}. \end{aligned}$$

Therefore

$$|y_i^*(x)| \le \frac{1}{2} \| x' \|_\epsilon + \frac{1}{\sqrt{3}} \| x'' \|_\pi, \le \frac{1}{\sqrt{3}} \| x \| .$$

However,

$$\begin{aligned} y_i^*(x) &= N \sum_{j=1}^N \lambda_i \int_0^1 f_{ij}^2 dt \\ &= N^2 \lambda_i \frac{1}{3N^2} = \frac{\lambda_i}{3}. \end{aligned}$$

Therefore

$$\| \sum_{i,j} \lambda_i f_{ij} \otimes e_{ij} \|_Z \ge \sqrt{3} \sup_i |y_i^*(x)|$$

$$\geq \frac{1}{\sqrt{3}} \sup |\lambda_i|.$$

Thus the space U is isomorphic to ℓ_∞^N, and we have the commuting diagram

$$\begin{tikzcd}
U \arrow[r, "T_{|U}"] \arrow[d, "A"'] & T(U) \\
\ell_\infty^N \arrow[r, "id_{\ell_\infty^N}"] & \ell_\infty^N \arrow[u, "A^{-1}"']
\end{tikzcd}$$

where $A: U \to \ell_\infty^N$ is the isomorphism between U and ℓ_∞^N. $\square$

3.4.5 Operators that factor through a Hilbert space.

It is well known that $\$(X,\ell_2) = \prod_2(X,\ell_2)$ whenever X is $C(K)$ or ℓ_1. One might ask whether this is true when $X = C(K,\ell_1)$. Indeed one could ask the weaker question: if $T: C(K,\ell_1) \longrightarrow \ell_2$ is bounded, does it follow that the induced operator $T^\#$ is 2-summing? We answer this question in the negative.

Theorem 3.4.9. *There is a compact Hausdorff space K and a bounded linear operator $T: C(K,\ell_1) \longrightarrow \ell_2$ for which $T^\#: C(K) \longrightarrow \prod_1(\ell_1,\ell_2)$ is not 2-summing.*

Proof. First, we show that there is a compact Hausdorff space K, and an operator $R: C(K) \longrightarrow \ell_\infty$ that is (2,1)-summing but not 2-summing. To see this, let $K = [0,1]$, and consider the natural embedding $C[0,1] \longrightarrow L_{2,1}[0,1]$, where $L_{2,1}[0,1]$ is the Lorentz space on $[0,1]$ with the Lebesque measure (see [6]). By [11], it follows that this map is (2,1)-summing. To show that this map is not 2-summing, we argue in a similar fashion to [8]. For $n \in \mathbb{N}$, consider the functions $e_i(t) = f(t + \frac{1}{i} \bmod 1)$ $(1 \leq i \leq n)$, where $f(t) = \frac{1}{\sqrt{t}}$ if $t \geq \frac{1}{n}$ and $\sqrt{n}$ otherwise. Then it is an easy matter to verify that for some constant $C > 0$,

$$\left(\sum_{i=1}^{n} |e^*(e_i)|^2\right)^{\frac{1}{2}} \leq C\sqrt{\log n}$$

for every e^* in the unit ball of $C[0,1]^*$, whereas

$$\left(\sum_{i=1}^{n} \|e_i\|^2_{L_{2,1}[0,1]}\right)^{\frac{1}{2}} \geq C^{-1} \log n.$$

Finally, since $L_{2,1}[0,1]$ is separable, it embeds isometrically into ℓ_∞.

Define $T: C(K,\ell_1) \to \ell_2$ as follows: for $\varphi = (f_n) \in C(K,\ell_1)$, let

$$T(f_n) = \sum_n Rf_n(n)e_n.$$

Then T is bounded, for

$$\| T(f_n) \|_2 = \left(\sum_n |Rf_n(n)|^2\right)^{\frac{1}{2}}$$

$$\leq \left(\sum_n \| Rf_n \|_{\ell_\infty}^2 \right)^{\frac{1}{2}}$$
$$\leq \pi_{2,1}(R) \sup_{t \in K} \sum_n |f_n(t)|.$$

Thus

$$\| T \| \leq \pi_{2,1}(R).$$

But $T^\# : C(K) \longrightarrow \$(\ell_1, \ell_2)$ is not 2-summing, because for each $f \in C(K)$, the operator $T^\# f : \ell_1 \longrightarrow \ell_2$ is the diagonal operator $\sum\limits_n Rf(n) e_n \otimes e_n$. Hence the strong operator norm of $T^\# f$ is

$$\| T^\# f \| = \sup_n |Rf(n)| = \| Rf \|_{\ell_\infty}.$$

Thus $T^\# : C(K) \longrightarrow \$(\ell_1, \ell_2)$ is not 2-summing, because $R : C(K) \longrightarrow \ell_\infty$ is not 2-summing.

□

3.4.6 Discussions and concluding remarks.

Remark 13

Remark 27. *Theorem 12 shows that if* X *and* Y *are Banach spaces such that* $\$(X, \ell_2) = \prod_2(X, \ell_2)$ *and* $\$(Y, \ell_2) = \prod_2(X, \ell_2)$*, then* $X \hat{\otimes}_\epsilon Y$ *need not share this property.*

This observation could also be deduced from arguments presented in [114] (use Example 3.5 and the proof of Proposition 3.6 to show that there is a bounded operator $T : (\ell_1 \oplus \ell_1 \oplus \ldots \oplus \ell_1)_{\ell_\infty} \longrightarrow \ell_2$ that is not p-summing for any $p < \infty$).

Remark 28. *In the proof of Theorem 2 we showed that the injective tensor product is an associative operation, that is, if* X, Y *and* Z *are Banach spaces, then* $(X \hat{\otimes}_\epsilon Y) \hat{\otimes}_\epsilon Z$ *is isometrically isomorphic to* $X \hat{\otimes}_\epsilon (Y \hat{\otimes}_\epsilon Z)$*. It is not hard to see that the same is true for the projective tensor product. However, we can conclude from Theorem 12 that what is known as the* γ_2^**-tensor product is not an associative operation.*

If E and F are Banach spaces, and $T : E \longrightarrow F$ is a bounded linear operator, following [10], we say that T **factors through a Hilbert space** if there is a Hilbert space H, and operators $B : E \longrightarrow H$ and $A : H \longrightarrow F$ such that $T = A \circ B$. We let $\gamma_2(T) = \inf\{\| A \| \| B \|\}$, where the infimum runs over all possible factorization of T, and denote the space of all operators $T : E \longrightarrow F$ that factor through a Hilbert space by $\Gamma_2(E, F)$. It is not hard to check that γ_2 defines a norm on $\Gamma_2(E, F)$, making $\Gamma_2(E, F)$ a Banach space. We define the γ_2^*-norm $\| \quad \|_*$ on $E \otimes F$ (see [108, 115]) in which the dual of $E \otimes F$ is identified with $\Gamma_2(E, F^*)$, and let $E \hat{\otimes}_{\gamma_2^*} F$ denote the completion of $(E \otimes F, \| \quad \|_*)$.

The operator $T : C(K) \hat{\otimes}_{\gamma_2^*} \ell_1 \longrightarrow \ell_2$ exhibited in Theorem 12, induces a bounded linear functional on $\left[(C(K) \hat{\otimes}_{\gamma_2^*} \ell_1) \hat{\otimes}_{\gamma_2^*} \ell_2 \right]^*$. Now we see that if $C(K) \hat{\otimes}_{\gamma_2^*} (\ell_1 \hat{\otimes}_{\gamma_2^*} \ell_2)$ were isometrically isomorphic to $(C(K) \hat{\otimes}_{\gamma_2^*} \ell_1) \hat{\otimes}_{\gamma_2^*} \ell_2$, then the operator $T^\# : C(K) \rightarrow \$(\ell_1, \ell_2)$ would induce a bounded linear functional on $\left[C(K) \hat{\otimes}_{\gamma_2^*} (\ell_1 \hat{\otimes}_{\gamma_2^*} \ell_2) \right]^*$, showing that $T^\# \in \Gamma_2 \left(C(K), \$(\ell_1, \ell_2) \right)$, implying that $T^\#$ would be 2-summing [10, p. 62]. This contradiction shows that $C(K) \hat{\otimes}_{\gamma_2^*} (\ell_1 \hat{\otimes}_{\gamma_2^*} \ell_2)$ and $\left(C(K) \hat{\otimes}_{\gamma_2^*} \ell_1 \right) \hat{\otimes}_{\gamma_2^*} \ell_2$ cannot be isometrically isomorphic. Another example showing that the γ_2^*-tensor product is not associative was given by Pisier (private communication).

3.5 Some deviation inequalities

Bernard Maurey
September 1990

Abstract:
We introduce a concentration property for probability measures on R^n, which we call Property (τ); we show that this property has an interesting stability under products and contractions (Lemmas 1, 2, 3). Using property (τ), we give a short proof for a recent deviation inequality due to Talagrand. In a third section, we also recover known concentration results for Gaussian measures using our approach.

3.5.1 Introduction.

Very roughly speaking, a concentration of measure phenomenon for a probability measure μ means that, given any measurable subset A such that $\mu(A) \geq \frac{1}{2}$, the enlargements of A, in a sense to be made precise, almost have measure 1. One important example of this situation is a consequence of Paul Lévy's isoperimetric inequality for the unit sphere S_n of R^{n+1}. This consequence is the following: Let μ_n denote the normalized rotation invariant measure on S_n; for every measurable subset A of S_n such that $\mu_n(A) \geq \frac{1}{2}$ we have

$$\mu_n\{x \in S_n; x \notin A_\varepsilon\} \leq \sqrt{\frac{\pi}{8}}\, e^{-n\varepsilon^2/2}$$

where A_ε denotes the subset of S_n of all points whose geodesic distance to A is less than ε (see [116]).

This result is crucial for the proof of Dvoretzky's theorem (about almost spherical sections of convex bodies) as given by V. Milman [117] (see also Figiel, Lindenstrauss and Milman [118]). Using Lévy's result, Borell [119] was able to prove an analogous isoperimetric result for the Gaussian measure on R^n (see also Ehrhard [120]). As it is the case for the sphere, Borell's isoperimetric result implies a Gaussian concentration of measure principle. Later, Pisier and the author gave a very simple proof for the Gaussian concentration principle which is needed for the proof of Dvoretzky's theorem (see [116, 121, 122]). Recently, Talagrand proved [123] a concentration principle for measures on R^n with exponential densities which is stronger than the Gaussian one (Corollary 1 below; this Corollary appears in [123] as a consequence of a more precise isoperimetric inequality which does not follow from our proof). We give here a proof for Talagrand's concentration result, using a property which we call Property (τ). This property is defined in section I, and general stability results about it are stated (Lemmas 1, 2, 3); we also explain in section I why property (τ) is related to concentration (Lemma 4). In section II, we show how to get Talagrand's inequality using property (τ). In section III we recover the Gaussian case along the same lines (Corollary 2). In a last section, we study a variant of property (τ), the *convex* property (τ); this variant is related to an other deviation inequality due to Talagrand [124] (see Corollary 5).

3.5.2 Property (τ).

Let f and g be two measurable functions on R^n; we denote by $f \square g$ the inf-convolution of f and g,

$$(f \square g)(x) = \inf\{f(x-y) + g(y); y \in R^n\}.$$

If μ is a probability measure on R^n and w a positive measurable function on R^n, we say that the couple (μ, w) satisfies the Property (τ) if for every bounded measurable function φ on R^n we have

$$(\int e^{\varphi \Box w} d\mu)(\int e^{-\varphi} d\mu) \leq 1.$$

If we adopt the convention $+\infty.0 \leq 1$, we see easily that the above inequality extends to all $\overline{R}$-valued measurable functions φ.

The definition of property (τ) was motivated by Talagrand's isoperimetric inequality for the cube, as the careful reader of [124] will notice.

Lemma 3.5.1. *If* (μ_i, w_i) *satisfies* (τ) *on* R^{n_i} *for* $i = 1, 2$, *then* $(\mu_1 \otimes \mu_2, w)$ *satisfies* (τ) *on* $R^{n_1} \times R^{n_2}$, *where*

$$w(x_1, x_2) = w_1(x_1) + w_2(x_2).$$

Proof. Consider $\varphi^y(x) = \varphi(x, y)$ and apply (τ) to $\psi(y) = \log(\int e^{\varphi^y \Box w_1} d\mu_1)$. □

Lemma 3.5.2. *If* (μ_i, w_i) *satisfies* (τ) *on* R^n *for* $i = 1, 2$, *then* $(\mu_1 * \mu_2, w_1 \Box w_2)$ *satisfies* (τ) *on* R^n.

Lemma 3.5.3. *Let* (μ_1, w_1) *satisfy* (τ) *on* R^{n_1}. *Let* w_2 *be a positive measurable function on* R^{n_2} *and* F *a mapping from* R^{n_1} *to* R^{n_2} *such that* $w_2(Fx - Fy) \leq w_1(x - y)$ *for every pair* x, y. *Let* μ_2 *be the image probability measure on* R^{n_2} *defined by* $\mu_2 = F(\mu_1)$. *Then* (μ_2, w_2) *satisfies* (τ).

Proof. $(\varphi \circ F) \Box w_1 \geq (\varphi \Box w_2) \circ F$.

Lipschitz maps like the above F were used by Pisier [121] in a slightly different context. Following him, we will be able in section III to pass from the Gaussian case to the uniform probability measure on $[0, 1]^n$ (Corollary 4).

□

We have perhaps to explain why property (τ) for a couple (μ, w) is a concentration of measure property.

Lemma 3.5.4. *Assume that* (μ, w) *satisfies* (τ) *on* R^n. *For every measurable subset* A *of* R^n *and every positive real number* t, *we have*

$$\mu\{x \notin A + \{w < t\}\} \leq (\mu(A))^{-1} e^{-t}.$$

Proof. Let A be a measurable subset of R^n and denote by φ_A the function equal to 0 on A and $+\infty$ outside. We observe that $(\varphi_A \Box w)(x) \geq t$ when $x \notin A + \{w < t\} = \{a + y; a \in A, w(y) < t\}$. Property (τ) implies that $\int e^{\varphi_A \Box w} d\mu \leq (\mu(A))^{-1}$ and we conclude with Tschebycheff's inequality. □

3.5.3 Talagrand's deviation inequality.

Let us define a function W on R by

$$W(t) = \frac{1}{18} t^2 \quad \text{for} \quad |t| \leq 2, \quad \frac{2}{9}(|t| - 1) \quad \text{otherwise}$$

and let μ_e be the probability measure on R with density $1_{(0,\infty)}(x) e^{-x}$.

Proposition 3.5.5. *The couple (μ_e, W) satisfies (τ).*

It follows from Lemma 2 that (ξ, U) also satisfies (τ), where ξ is the convolution of μ_e and its symmetric image on $(-\infty, 0)$, and $U = W \Box W$. It is easy to see that ξ has density $\frac{1}{2}e^{-|x|}$ on R, and that

$$U(t) = 2W(t/2) = \frac{1}{36}t^2 \quad \text{for} \quad |t| \le 4, \quad \frac{2}{9}(|t| - 2) \quad \text{otherwise.}$$

We deduce now from Lemma 1 that the couple (ξ_n, U_n) satisfies (τ) on R^n for every n, where ξ_n is the product of n copies of ξ and $U_n(x) = \sum_{i=1}^n U(x_i)$. The idea of working with functions like W or U_n comes from Talagrand [123].

Theorem 3.5.6. *The couple (ξ_n, U_n) satisfies (τ) for every integer n. In particular, for every measurable subset A of R^n we have if we set $\rho_A = \varphi_A \Box U_n$*

$$\int e^{\rho_A} d\xi_n \le (\xi_n(A))^{-1}.$$

Corollary 3.5.7 (Talagrand). *For every $t > 0$,*

$$\xi_n\{x; x \notin A + 6\sqrt{t}B_2 + 9tB_1\} \le (\xi_n(A))^{-1}e^{-t}.$$

where B_2 and B_1 are respectively the usual ℓ_2^n and ℓ_1^n balls.

Proof. According to Lemma 4, we need only show that

$$\{U_n < t\} \subset 6\sqrt{t}B_2 + 9tB_1.$$

Assume $U_n(x) < t$, and define y and z in the following way: $y_i = x_i$ if $|x_i| \le 4$, $y_i = 0$ otherwise; $z_i = x_i$ if $|x_i| > 4$, $z_i = 0$ otherwise. Then $x = y + z$ and it is easy to check that $\|y\|_2 \le 6\sqrt{t}$, $\|z\|_1 \le 9t$. $\Box$

We present now the proof of the above Proposition. Let φ be a bounded measurable function on $(0, +\infty)$, and let ψ denote the function $\varphi \Box W$.

Let $I_0 = \int_0^\infty e^{-\varphi(x)-x}dx$ and $I_1 = \int_0^\infty e^{\psi(y)-y}dy$. For $t \in (0,1)$, we define $x(t)$ and $y(t)$ by the relations

$$\int_0^{x(t)} e^{-\varphi(x)-x}dx = tI_0, \int_0^{y(t)} e^{\psi(y)-y}dy = tI_1.$$

We obtain by differentiation

$$x'(t) = I_0 e^{\varphi(x(t))+x(t)}, y'(t) = I_1 e^{-\psi(y(t))+y(t)}.$$

Taking into account the fact that $\psi(y(t)) \le \varphi(x(t)) + W(x(t) - y(t))$, we obtain

$$y'(t) \ge I_1 e^{-\varphi(x(t))-W(x(t)-y(t))+y(t)}.$$

Let now $z(t) = \frac{1}{2}(x(t) + y(t)) - W(x(t) - y(t))$. We have

$$z'(t) = \big(\frac{1}{2} - W'(x(t) - y(t))\big)x'(t) + \big(\frac{1}{2} + W'(x(t) - y(t))\big)y'(t).$$

If we use the fact that $|W'| \leq 1/2$ on R, we get (writing x and y for $x(t)$ and $y(t)$ for simplicity, and using the inequality $\frac{1}{2}(ua + v/a) \geq \sqrt{uv}$ with $a = e^{\varphi(x)}$)

$$
\begin{aligned}
z'(t) &\geq (1 - 2W'(x-y))I_0 e^x \,\frac{e^{\varphi(x)}}{2} + (1 + 2W'(x-y))I_1 e^{-W(x-y)+y}\,\frac{e^{-\varphi(x)}}{2} \\
&\geq \sqrt{1 - 4W'(x-y)^2}\,\sqrt{I_0 I_1}\, e^{\frac{1}{2}(x+y) - \frac{1}{2}W(x-y)} \\
&= \sqrt{I_0 I_1}\, e^{z(t)} \sqrt{1 - 4W'(x-y)^2}\, e^{\frac{1}{2}W(x-y)}.
\end{aligned}
$$

and this is our Proposition.

Claim: We claim that for every s

$$(1 - 4W'(s)^2)\, e^{W(s)} \geq 1.$$

It will then follow that $e^{-z(t)} z'(t) \geq \sqrt{I_0 I_1}$, which yields after integrating between 0 and 1

$$1 \geq \sqrt{I_0 I_1}$$

.

Proof of the claim: We only consider $s \geq 0$ since W is even. For $s \geq 2$, W' is constant and W increasing, so it is enough to check the case $0 \leq s \leq 2$; this reduces to

$$e^{-u/18} \leq 1 - 4u/81 \quad \text{for} \quad u \in (0, 4)$$

which is proved using elementary calculus. ☐

3.5.4 The Gaussian case.

Let γ be the standard Gaussian probability measure on R, with density $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, and γ_n the product of n copies of γ. Throughout this section, the norm will be the Euclidean norm on R^n.

Theorem 3.5.8. *The couple* $(\gamma_n, \frac{1}{4}\|x\|^2)$ *satisfies* (τ) *for every integer* n.

Proof. We check first that $(\gamma, x^2/4)$ has property (τ) on R; the proof is similar to the proof of the Proposition, but simpler: $x(t)$ and $y(t)$ are defined in a similar fashion, and $z(t)$ is simply equal to $\frac{1}{2}(x(t) + y(t))$. It follows from Lemma 1 that $(\gamma_n, \frac{1}{4}\|x\|^2)$ has property (τ) for every integer n.

We can also give a direct proof using the functional Brunn-Minkowski inequality due to Prekopa and Leindler [122, 125, 126]: If f, g, h are bounded below measurable functions on R^n such that for all x and u we have $\frac{1}{2}\big(f(x+u) + g(x-u)\big) \geq h(x)$, then

$$\Big(\int e^{-f(x)} dx\Big)\Big(\int e^{-g(x)} dx\Big) \leq \Big(\int e^{-h(x)} dx\Big)^2.$$

We apply this inequality to $f(x) = \varphi(x) + \frac{1}{2}\|x\|^2, g(y) = -\psi(y) + \frac{1}{2}\|y\|^2$ and $h(z) = \frac{1}{2}\|z\|^2$, where we have set $\psi = \varphi \Box w, w(y) = \frac{1}{4}\|y\|^2$. ☐

Remark 29. *The second proof of Theorem 2 only uses the uniform convexity properties of* $-\log f$*, where* f *is the density of* μ*.*

Remark 30. *As pointed out by Talagrand, the Gaussian concentration result is a consequence of Corollary 1, using a suitable map that transforms* ξ_n *into* γ_n*. More precisely, Lemma 3 and Theorem 1 imply that for some* $a > 0$*, the couple* $(\gamma_n, a\|x\|^2)$ *satisfies* (τ) *for every* n*. However, the proof of Theorem 2 gives a better constant* a *and is simpler.*

We will show now that Theorem 2 allows to recover the main conclusion of the Gaussian concentration result of [121].

Corollary 3.5.9. *Let* φ *be a 1-Lipschitz function on* R^n*, and* X, Y *two independent* n*-dimensional Gaussian vectors with distribution equal to* γ_n*. For every real number* λ *we have*

$$Ee^{\frac{\lambda}{\sqrt{2}}(\varphi(X)-\varphi(Y))} \leq e^{\lambda^2/2}.$$

Proof. Let $\psi = \frac{\lambda\varphi}{\sqrt{2}} \Box w$, where φ is 1-Lipschitz on R^n, $w(y) = \frac{1}{4}\|y\|^2$ and $\lambda > 0$. It is enough to apply (τ) and notice that $\psi(x) \geq \frac{\lambda\varphi}{\sqrt{2}}(x) - \lambda^2/2$. Let y be such that

$$\psi(x) = \frac{\lambda\varphi}{\sqrt{2}}(y) + \frac{1}{4}\|x-y\|^2.$$

Then

$$\psi(x) \geq \frac{\lambda\varphi}{\sqrt{2}}(x) - \frac{\lambda}{\sqrt{2}}\|x-y\| + \frac{1}{4}\|x-y\|^2 \geq \frac{\lambda\varphi}{\sqrt{2}}(x) + \min\{\frac{1}{4}u^2 - \frac{\lambda}{\sqrt{2}}u; u \in R\} = \frac{\lambda\varphi}{\sqrt{2}}(x) - \lambda^2/2.$$

□

The first part of the next Corollary is known (it is a Poincaré-type inequality due to Chen [127]).

Corollary 3.5.10. *If* φ *is a Lipschitz function on* R^n*, we have*

$$\frac{1}{2}\int (\varphi(x) - \varphi(y))^2 d\gamma_n(x) d\gamma_n(y) \leq \int \|\nabla\varphi\|^2 d\gamma_n.$$

More generally, this result holds for every probability measure μ *on* R^n *such that* (μ, w) *satisfies* (τ) *for a function* w *convex and greater than* $\frac{1}{4}\|x\|^2$ *in a neighborhood of* 0*.*

Proof. Let u be a convex function such that $u \leq w$ and $u(x) = \frac{1}{4}\|x\|^2$ in a neighborhood of 0; assume that φ is a compactly supported C^1-function. For $t > 0$ consider $\varphi_t = t\varphi$ and $\psi_t = \varphi_t \Box u$. One can check that

$$\lim_{t\to 0} \frac{\psi_t(x) - \varphi_t(x)}{t^2} = -\|\nabla\varphi(x)\|^2$$

and the result follows easily from the property (τ) of (μ, u) applied to φ_t, when $t \to 0$. □

Remark 31. *If* (μ, w) *satisfies* (τ) *on* R*, where* μ *is such that* $\frac{1}{2}\int (x-y)^2 d\mu(x) d\mu(y) > 1$*, we can apply Corollary 3 to* $\varphi(x) = x$ *to conclude that* $\{x; w(x) \geq \frac{1}{4}x^2\}$ *is not a neighborhood of* 0*. One can also show that if* (μ, w) *satisfies* (τ) *on* R*, with* μ *symmetric and* $\int x^2 d\mu(x) = 1$*, then* $(\gamma, \frac{1}{2}w''(0)t^2)$ *also satisfies* (τ)*. This shows the necessity of a subquadratic behavior at* 0 *for the function* w*.*

Corollary 3.5.11. *Let* λ_n *denote the uniform probability measure on* $[0,1]^n$*. There exists* $a > 0$ *such that* $(\lambda_n, a\|x\|^2)$ *satisfies* (τ) *for every integer* n *(one can take* $a = \pi/2$*).*

Proof. Using Lemma 3, this follows from Theorem 2, exactly like in Pisier [121]. □

3.5.5 Convex property (τ).

Assume that w is a convex function on some topological vector space X and that μ is a probability measure on X. We say that the couple (μ, w) satisfies the *convex* property (τ) provided

$$(\int e^{\varphi \Box w} d\mu)(\int e^{-\varphi} d\mu) \leq 1$$

for every *convex* measurable function φ on X.

Lemma 3.5.12. *If (μ_i, w_i) satisfies the convex property (τ) on X_i for $i = 1, 2$, then $(\mu_1 \otimes \mu_2, w)$ satisfies the convex property (τ) on $X_1 \times X_2$, with*

$$w(x_1, x_2) = w_1(x_1) + w_2(x_2).$$

Proof: As in the proof of Lemma 1 we consider $\varphi^y(x) = \varphi(x, y)$ for a convex function φ on $X_1 \times X_2$; we can apply the convex property (τ) to $\psi(y) = \log(\int e^{\varphi^y \Box w_1} d\mu_1)$ if we observe that ψ is a convex function.

We shall say that μ has diameter ≤ 1 as a short way to express that μ is supported by a set of diameter ≤ 1. The following Theorem is the equivalent in our language of a result of Talagrand [124] and its generalization by Johnson and Schechtman [128].

Theorem 3.5.13. *Let (X_i) be a family of normed spaces; for each i, let μ_i be a probability measure with diameter ≤ 1 on X_i, and $w_i(x) = \frac{1}{4}\|x\|^2$ for $x \in X_i$. If μ is the product of the family (μ_i), then (μ, w) satisfies the convex property (τ), with $w(x) = \sum_i w_i(x_i)$.*

Proof. According to Lemma 5, we only need to prove the result for a single probability measure μ with diameter ≤ 1 on a normed space X. Let A be a set of diameter ≤ 1 that supports μ, and let φ be a convex function on X; assume without loss of generality that $\inf \varphi(A) = 0$. Define $w(x) = \frac{1}{4}\|x\|^2$ and $\psi = \varphi \Box w$. Let $x \in A$, $\varepsilon > 0$ and $a \in A$ such that $\varphi(a) \leq \varepsilon$. We have, if $y = (1-\theta)x + \theta a$ and $0 \leq \theta \leq 1$

$$\psi(x) \leq \varphi(y) + \frac{1}{4}\|x - y\|^2 \leq (1-\theta)\varphi(x) + \theta\varepsilon + \frac{1}{4}\theta^2.$$

Choosing an optimal θ, we deduce from the above that $\psi(x) \leq k(\varphi(x))$ where $k(u)$ is equal to $u - u^2$ if $0 \leq u \leq \frac{1}{2}$, and to $\frac{1}{4}$ if $u \geq \frac{1}{2}$. We claim now that $e^{k(u)} \leq 2 - e^{-u}$. It follows that

$$\int e^{\psi} d\mu \leq 2 - \int e^{-\varphi} d\mu \leq (\int e^{-\varphi} d\mu)^{-1}$$

and this finishes the proof (the preceding computation was inspired by [128]). □

Proof of the claim: For $0 \leq u \leq \frac{1}{2}$, we write

$$\frac{1}{2}(e^{u-u^2} + e^{-u}) = e^{-u^2/2}\cosh(u - u^2/2) \leq e^{-u^2/2}\cosh(u) \leq 1.$$

□

Remark 32. *In the case of the probability β on $[0, 1]$ that gives measure $\frac{1}{2}$ to $\{0\}$ and $\{1\}$, it is easy to improve the function w from $\frac{1}{4}x^2$ to $\frac{1}{2}x^2$.*

Corollary 3.5.14. *[124, 128] Let A be a measurable subset of $[0,1]^n$ and B its convex hull. For every product probability measure μ on $[0,1]^n$ we have*

$$\int e^{\frac{1}{4}d_B^2}\, d\mu \leq (\mu(A))^{-1}$$

where d_B denotes the Euclidean distance to the set B.

3.6 Ellipsoids of maximal volume in convex bodies

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Abstract.
The largest discs contained in a regular tetrahedron lie in its faces. The proof is closely related to the theorem of Fritz John characterising ellipsoids of maximal volume contained in convex bodies.

3.6.1 Introduction.

In 1948, Fritz John proved that each convex body in $\mathbb{R}^n$ contains an unique ellipsoid of maximal volume. Thus, each convex body has an affine image whose ellipsoid of maximal volume is the Euclidean unit ball, B_2^n. John characterised these affine images with the following theorem.

Theorem 3.6.1. *[129]. The Euclidean ball is the ellipsoid of maximal volume contained in the convex body $C \subset \mathbb{R}^n$ if and only if $B_2^n \subset C$ and, for some $m \geq n$, there are Euclidean unit vectors $(u_i)^m$, on the boundary of C, and positive numbers $(c_i)_1^m$ for which*

a) $\sum_i c_i u_i = 0$ *and*

b) $\sum_i c_i u_i \otimes u_i = I_n$*, the identity on* $\mathbb{R}^n$*.* □

The u_i's of the theorem, are points of contact of the unit sphere S^{n-1} with the boundary of C. The theorem says that weights may be distributed on the collection of such points so that, a) the centre of mass of the distribution is at the origin and b) the inertia tensor of the distribution is the identity. The first condition shows that the contact points do not all lie "on one side" of the sphere, and the second, that they do not all lie "close to proper subspace".

Condition b) shows that the u_i's behave like an orthonormal basis in that the inner product can be represented

$$\langle x, y \rangle = \sum_i c_i \langle u_i, x \rangle \langle u_i, y \rangle, \quad x, y \in \mathbb{R}^n.$$

It follows immediately from the equality of the traces of the operators in b) that

$$\sum_i c_i = n. \tag{1}$$

At each u_i, the supporting hyperplane to C (is unique and) is perpendicular to u_i (since this is true for the Euclidean ball). Hence, the set $K = \{x \in \mathbb{R}^n \colon \ \langle x, u_i \rangle \leq 1, \ 1 \leq i \leq m\}$ contains $C : C$ is sandwiched between B_2^n and K. From this it follows (and this was one of the motivations for John's theorem) that C is contained in nB_2^n, the Euclidean ball of radius n. To see this, suppose $x \in K$ and $\|x\| = r$. Since $-r \leq \langle x, u_i \rangle \leq 1$ for every i,

$$\begin{aligned}0 &\leq \sum_i c_i(1-\langle x,u_i\rangle)(r+\langle x,u_i\rangle)\\ &= r\sum_i c_i + (1-r)\sum_i c_i\langle x,u_i\rangle - \sum_i c_i\langle x,u_i\rangle^2.\end{aligned}$$

Properties a) and b) and (1) show that the latter is

$$rn - \|x\|^2 = rn - r^2.$$

Hence $r \leq n$.

John's theorem has been used many times in the theory of finite-dimensional normed spaces. For symmetric convex bodies (the unit balls of normed spaces) condition a) is redundant and a stronger estimate $\|x\| \leq \sqrt{n}$ can be obtained for $x \in K$. As a consequence, every n-dimensional normed space is isomorphic, with isomorphism constant at most $\sqrt{n}$, to n-dimensional Euclidean space. It was pointed out to me a few months ago by Prof. A. Pełcyński that the literature does not contain any very explicit proof of the easy (if) part of John's theorem. The first section of this article contains a quick proof of this assertion. At about the same time, several people (not quite independently) asked me for a proof of the intuitively obvious fact that the largest discs contained in a regular tetrahedron, lie in its faces. The second section of this article consists of a proof of (the analogue of) this fact for k-dimensional Euclidean balls inside regular n-dimensional simplices. For each n and $k < n$, the regular n-dimensional simplex which circumscribes B_2^n, contains a k-dimensional Euclidean ball of radius

$$\sqrt{\frac{n(n+1)}{k(k+1)}} = r(n,k) \quad \text{(say)},$$

in each of its k-dimensional faces. The relationship between Sections 1 and 2 of this article is elucidated in Section 3 where it is shown that if C is a convex body in $\mathbb{R}^n$ whose ellipsoid of maximal volume is B_2^n, then C does not contain k-dimensional ellipsoids whose volume is larger than that of a k-dimensional ball of radius $r(n,k)$. The result of Section 3, and, a fortiori, that of Section 2, could certainly be "checked by hand". It is enough to show that the convex hull of B_2^n and a "large" k-dimensional ellipsoid, contains an n-dimensional ellipsoid of volume larger than B_2^n. But the calculations involved are messy. The argument presented in Section 3 is a compelling, if simple, illustration of the usefulness of John's characterisation.

3.6.2 The 'if' part of John's theorem.

Proposition 3.6.2. *Let* $(u_i)_1^m$ *be a sequence of unit vectors in* $\mathbb{R}^n$ *and* $(c_i)_1^m$ *a sequence of positive numbers satisfying*

a) $\sum_i c_i u_i = 0$ *and*

b) $\sum_i c_i u_i \otimes u_i = I_n$.

Then the set $K = \{x \in \mathbb{R}^n\colon\ \langle x,u_i\rangle \leq 1,\ 1 \leq i \leq m\}$ *contains an unique ellipsoid of maximal volume, the Euclidean unit ball.*

Proof. Let $\mathcal{E}$ be the ellipsoid,

$$\left\{ x \in \mathbb{R}^n \colon \ \sum_1^n \alpha_j^{-2} \langle x - y, v_j \rangle^2 \leq 1 \right\}$$

for some $y \in \mathbb{R}^n$, orthonormal basis $(v_j)_1^n$ and positive numbers $(\alpha_j)_1^n$. The problem is to show that if $\mathcal{E} \subset K$, then $\Pi \alpha_j \leq 1$ with equality only if $\alpha_j = 1$ for all j, and $y = 0$. Now, for each i, $1 \leq i \leq m$, the point

$$x_i = y + \left(\sum_{j=1}^n \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{-\frac{1}{2}} \sum_{j=1}^n \alpha_j^2 \langle u_i, v_j \rangle v_j$$

belongs to $\mathcal{E}$ and so $\langle u_i, x_i \rangle \leq 1$ for each i. Hence

$$\langle u_i, y \rangle + \left(\sum_{j=1}^n \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq 1 \tag{2}$$

for each i. Multiply by c_i, sum over i and use the fact that $\sum\limits_i c_i u_i = 0$ to get

$$\sum_i c_i \left(\sum_j \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq \sum_i c_i = n.$$

Since $\sum\limits_i c_i \langle u_i, x \rangle^2 = \|x\|^2$ for all x, and the v_j's form an orthonormal basis,

$$\begin{aligned}
\sum_j \alpha_j &= \sum_j \sum_i \alpha_j c_i \langle u_i, v_j \rangle^2 \\
&= \sum_i c_i \left(\sum_j \alpha_j \langle u_i, v_j \rangle^2 \right) \\
&\leq \sum_i c_i \left(\sum_j \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \left(\sum_j \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \\
&= \sum_i c_i \left(\sum_j \alpha_j^2 \langle u_i, v_j \rangle^2 \right)^{\frac{1}{2}} \leq n.
\end{aligned}$$

By the $AM \backslash GM$ inequality $\Pi \alpha_j \leq 1$. There is equality only if $\alpha_j = 1$ for all j in which case (2) says that

$$\langle u_i, y \rangle + \|u_i\| \leq 1 \quad \text{for all} \quad i,$$

i.e. $\langle u_i, y \rangle \leq 0$ for all i. Since $\sum c_i \langle u_i, y \rangle = 0$, this implies that $\langle u_i, y \rangle = 0$ for all i and so $y = 0$. □

3.6.3 The simplex.

Proposition 3.6.3. *Let T be a regular solid simplex in $\mathbb{R}^n$ of internal radius 1. For $1 \leq k \leq n-1$, the largest k-dimensional Euclidean balls in T are those of radius*

$$\sqrt{\frac{n(n+1)}{k(k+1)}}$$

which lie in k-dimensional faces of T.

The proof of Propositon 2 makes use of the following well-known application of Caratheodory's theorem.

Lemma 3.6.4. *If $(x_i)_1^m$ is a sequence in $\mathbb{R}^k$ of diameter at most d, there is a point $x \in \mathbb{R}^k$ with*

$$\|x_i - x\| \leq r = d\sqrt{\frac{k}{2(k+1)}} \quad \textit{for all} \quad i.$$

The bound is sharp only if $(x_i)_1^m$ includes some $k+1$ points, all at a distance r from their average.

Proof. Let x be the point of $\mathbb{R}^k$ which minimises $\max_i \|x_i - x\|$ and suppose that this maximum is s. Than x is in the convex hull of those x_i's from which it has distance s since otherwise there would be a small perturbation, y, of x with $\max_i \|x_i - y\| < s$. By Caratheodory's theorem, x is a convex combination of some $k+1$ of the x_i's at distance s from x: say

$$x = \sum_1^{k+1} \lambda_i x_i.$$

Then

$$\begin{aligned} s^2 &= \sum_1^{k+1} \lambda_i \|x_i - x\|^2 \\ &= \frac{1}{2} \sum_{i,j} \lambda_i \lambda_j \|x_i - x_j\|^2 \\ &= \frac{1}{2} \sum_{i \neq j} \lambda_i \lambda_j \|x_i - x_j\|^2 \\ &\leq \frac{1}{2} d^2 \sum_{i \neq j} \lambda_i \lambda_j = \frac{1}{2} d^2 \left(\left(\sum \lambda_i \right)^2 - \sum \lambda_i^2 \right) \\ &\leq \frac{d^2 k}{2(k+1)} \end{aligned}$$

since $\sum \lambda_i^2 \geq \frac{1}{k+1} \left(\sum \lambda_i \right)^2 = \frac{1}{k+1}$ by the Cauchy-Schwartz inequality.

For the estimate to be sharp, one needs that $\lambda_i = \frac{1}{k+1}$, $1 \leq i \leq k+1$, implying the second assertion of the lemma. □ □

Proof of Proposition 2. Assume that T is given by

$$T = \{x \in \mathbb{R}^n \colon\ \langle x, u_i\rangle \le 1,\ 1 \le i \le n+1\}$$

for an appropriate sequence $(u_i)_1^{n+1}$ of unit vectors and note that

$$\sum_i u_i = 0 \tag{3}$$

$$\sum_i u_i \otimes u_i = \frac{n+1}{n} I_n. \tag{4}$$

Let $\mathcal{E} = \{x \in \mathbb{R}^n \colon\ \sum_1^k \langle x-y, v_j\rangle^2 \le r^2, \langle x-y, v_j\rangle = 0, k+1 \le j \le n\}$ be a k-dimensional ball contained in T, for some $y \in \mathbb{R}^n$, $r > 0$ and orthonormal basis $(v_j)_1^n$. As in the proof of Proposition 1,

$$\langle u_i, y\rangle + r\left(\sum_{j=1}^k \langle u_i, v_j\rangle^2\right)^{\frac{1}{2}} \le 1 \quad \text{for all} \quad i.$$

Let P be the orthogonal projection of $\mathbb{R}^n$ onto $\operatorname{span}(v_j)_1^k$. Then for all i

$$\langle u_i, y\rangle + r\|Pu_i\| \le 1. \tag{5}$$

Summing over i and using (3),

$$r \sum_i \|Pu_i\| \le n+1$$

so it is enough to show that

$$\sum_i \|Pu_i\| \ge \sqrt{\frac{k(k+1)(n+1)}{n}} \tag{6}$$

for every orthogonal projection of rank k.

Now, the set $\{u_i\}_1^{n+1}$ has diameter

$$\sqrt{\frac{2(n+1)}{n}}$$

and hence the set $\{Pu_i\}_1^{n+1}$ has diameter at most this. Since $\{Pu_i\}_1^{n+1}$ sits in an Euclidean space of dimension k, Lemma 3 shows that there is a point $x \in P(\mathbb{R}^n)$ with

$$\|Pu_i - x\| \leq \sqrt{\frac{k(n+1)}{(k+1)n}} \quad \text{for all} \quad i. \tag{7}$$

From identity (4),

$$\sum_i Pu_i \otimes Pu_i = \frac{n+1}{n} P$$

and equating traces,

$$\sum_i \|Pu_i\|^2 = \frac{n+1}{n} k.$$

Since $\sum\limits_i Pu_i = 0$,

$$\begin{aligned}
\frac{n+1}{n} k &= \sum_i \langle Pu_i, Pu_i \rangle \\
&= \sum_i \langle Pu_i, Pu_i - x \rangle \\
&\leq \sum_i \|Pu_i\| \cdot \|Pu_i - x\| \\
&\leq \sqrt{\frac{k(n+1)}{(k+1)n}} \sum_i \|Pu_i\|,
\end{aligned}$$

giving the desired inequality (6).

Now, suppose the maximum radius is attained. Then there is equality in (7) so, by Lemma 3 again, the set $\{Pu_i\}$ includes $k+1$ points all at distance $\sqrt{\frac{k(n+1)}{(k+1)n}}$ from their average. But every $k+1$ of the u_i's are at this distance from their average. So P is an isometry on the affine hull of some $k+1$, u_i's: i.e. the range of P is parallel to this affine hull. This implies that $\mathcal{E}$ lies in a k-dimensional subspace parallel to some k-dimensional face of T. This fact determines all the numbers $\|Pu_i\|$ and hence the numbers $\langle u_i, y \rangle$ since there is equality in (5) for all i. These numbers determine y. □

3.6.4 The general case.

Proposition 3.6.5. *Let $(u_i)_1^m$, $(c_i)_1^m$ and K be as in Proposition 1. If $\mathcal{E}$ is a k-dimensional ellipsoid in K then the (k-dimensional) volume of $\mathcal{E}$ is no larger than that of a k-dimensional ball of radius $\sqrt{\frac{n(n+1)}{k(k+1)}}$.*

This proposition cannot be proved using the argument of Proposition 2 as it stands, since for a general sequence $(u_i)_1^m$ satisfying the hypotheses and orthogonal projection P of rank k, $\sum\limits_i c_i \|Pu_i\|$ may be as small as k. This complicates the argument somewhat: Proposition 2 is isolated because it has a simpler proof. For the proof of Proposition 4, Lemma 3 is replaced by an easier observation.

Lemma 3.6.6. *If $(x_i)_1^m$ is a sequence of vectors with $\sum x_i = 0$ and $(u_i)_1^m$, a sequence of unit vectors in some Euclidean space then*

$$\left(\sum_i \langle x_i, u_i\rangle\right)^2 \leq \sum_{i,j} \|x_i\| \cdot \|x_j\|(1 - \langle u_i, u_j\rangle).$$

Proof. By homogeneity, it may be assumed that $\sum\limits_i \|x_i\| = 1$. Set $\lambda_i = \|x_i\|$, $1 \leq i \leq m$ and $u = \sum\limits_i \lambda_i u_i$. Then,

$$\begin{aligned}\left(\sum_i \langle x_i, u_i\rangle\right)^2 &= \left(\sum \langle x_i, u_i - u\rangle\right)^2 \\ &\leq \left(\sum \|x_i\| \cdot \|u_i - u\|\right)^2 \\ &= \left(\sum \lambda_i \|u_i - u\|\right)^2 \\ &\leq \sum \lambda_i \|u_i - u\|^2 = 1 - \|u\|^2 \\ &= \sum_{i,j} \lambda_i \lambda_j (1 - \langle u_i, u_j\rangle). \qquad \square\end{aligned}$$

□

Proof of Proposition 4. Let $\mathcal{E}$ be the ellipsoid

$$\left\{x \in \mathbb{R}^n : \ \sum_1^k \alpha_j^{-2} \langle x - y, v_j\rangle^2 \leq 1, \right.$$
$$\left. \langle x - y, v_j\rangle = 0, \ k+1 \leq j \leq n\right\}$$

for some $y \in \mathbb{R}^n$, orthonormal basis $(v_j)_1^n$ and positive numbers $(\alpha_j)_1^k$. The problem is to show that

$$\left(\prod_1^k \alpha_j\right)^{\frac{1}{k}} \leq \sqrt{\frac{n(n+1)}{k(k+1)}}.$$

It certainly suffices to show that

$$\sum_1^k \alpha_j \leq \sqrt{\frac{kn(n+1)}{k+1}}.$$

As in the proof of Proposition 1,

$$\langle u_i, y\rangle + \left(\sum_{j=1}^{k} \alpha_j^2 \langle u_i, v_j\rangle^2\right)^{\frac{1}{2}} \leq 1$$

for every i. Define $T\colon\ \mathbb{R}^n \to \mathbb{R}^n$ by

$$Tx = \sum_{j=1}^{k} \alpha_j \langle x, v_j\rangle v_j.$$

Then

$$\langle u_i, y\rangle + \|Tu_i\| \leq 1 \quad \text{for every} \quad i. \tag{8}$$

Also

$$\sum_{1}^{k} \alpha_j^2 = \sum_{i,j} \alpha_j^2 c_i \langle u_i, v_j\rangle^2 = \sum_i c_i \|Tu_i\|^2 \tag{9}$$

and

$$\sum_{1}^{k} \alpha_j = \sum_{i,j} \alpha_j c_i \langle u_i, v_j\rangle^2 = \sum_i c_i \langle Tu_i, u_i\rangle. \tag{10}$$

The proof divides into two parts, the first of which effectively handles the u_i's which are far apart (as if the body were symmetric) while the second, and more complicated part, handles the u_i's which are close together (as if the body were a simplex).

Since $\|Tu_i\| \leq 1 - \langle u_i, y\rangle$ for each i (by (8))

$$\begin{aligned}\sum_i c_i \|Tu_i\|^2 &\leq \sum_i c_i (1 - \langle u_i, y\rangle)^2 \\ &= \sum_i c_i - \left\langle \sum_i c_i u_i, y\right\rangle + \sum_i c_i \langle u_i, y\rangle^2 \\ &= n + \|y\|^2.\end{aligned}$$

So by (9)

$$\frac{1}{k}\left(\sum_i \alpha_j\right)^2 \leq \sum_j \alpha_j^2 \leq n + \|y\|^2. \tag{11}$$

On the other hand, set $x_i = c_i T u_i,\ 1 \leq i \leq m$ and observe that $\sum x_i = T\big(\sum c_i u_i\big) = 0$. Then Lemma 5 and (10) show that

$$\begin{aligned}\left(\sum_1^k \alpha_j\right)^2 &= \left(\sum_i \langle x_i, u_i\rangle\right)^2\\ &\le \sum_{i,j} \|x_i\| \cdot \|x_j\|(1-\langle u_i, u_j\rangle)\\ &= \sum_{i,j} c_i c_j (1-\langle u_i, u_j\rangle)\|Tu_i\| \cdot \|Tu_j\|.\end{aligned}$$

Since $c_i c_j(1-\langle u_i, u_j\rangle) \ge 0$ for all i and j, (8) can be applied again to give

$$\left(\sum \alpha_j\right)^2 \le \sum_{i,j} c_i c_j (1-\langle u_i, u_j\rangle)(1-\langle u_i, y\rangle)(1-\langle u_j, y\rangle).$$

Expanding this product and using the fact that $\sum\limits_i c_i u_i = 0$ one obtains

$$\left(\sum \alpha_j\right)^2 \le \left(\sum c_i\right)^2 - \sum_{i,j} c_i c_j \langle u_i, u_j\rangle\langle u_i, y\rangle\langle u_j, y\rangle.$$

Two applications of the identity $\sum\limits_i c_i\langle u_i, x\rangle\langle u_i, y\rangle = \langle x, y\rangle$ show that

$$\left(\sum \alpha_j\right)^2 \le n^2 - \|y\|^2. \tag{12}$$

Finally, this inequality may be added to (11) to give

$$\left(1+\frac{1}{k}\right)\left(\sum \alpha_j\right)^2 \le n^2 + n,$$

and hence

$$\left(\sum \alpha_j\right)^2 \le \frac{kn(n+1)}{k+1}$$

as required. □

Remark 33. *For $k = 1$, Proposition 4 states that if C is a convex body whose ellipsoid of maximal volume is B_2^n then $\operatorname{diam}(C) \le \sqrt{2n(n+1)}$. This fact can be proved more simply: if $x, y \in C$ then $\|x\|, \|y\| \le n$, as explained in the introduction, and (with the usual notation)*

$$\begin{aligned}0 &\le \sum_i c_i(1-\langle u_i, x\rangle)(1-\langle u_i, y\rangle)\\ &= n + \langle x, y\rangle\end{aligned}$$

so that

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$$

$$\leq 2n^2 + 2n = 2n(n+1).$$

The fact that $\|x\| \leq n$ *for* $x \in C$ *could be deduced from inequality (12) of the above proof, with "*$k = 0$*".*

3.7 The plank problem for symmetric bodies

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(1)Partially supported by NSF DMS-8807243

Abstract.
Given a symmetric convex body C and n hyperplanes in an Euclidean space, there is a translate of a multiple of C, at least $\frac{1}{n+1}$ times as large, inside C, whose interior does not meet any of the hyperplanes. The result generalizes Bang's solution of the plank problem of Tarski and has applications to Diophantine approximation.

AMS :1980 Subject classification: 52A37, 46A22

3.7.1 Introduction and preliminary observations.

In the 1930's, Tarski posed what came to be known as the plank problem. A *plank* in $\mathbb{R}^d$ is the region between two distinct parallel hyperplanes. Tarski conjectured that if a convex body of minimum width w is covered by a collection of planks in $\mathbb{R}^d$, then the sum of the widths of these planks is at least w. Tarski himself proved this for the disc in $\mathbb{R}^2$. The problem was solved in general by Bang [130]. At the end of his paper, Bang asked whether his theroem could be strengthened by asking that the width of each plank should be measured relative to the width of the convex body being covered, in the direction of the normal to the plank. This affine invariant plank problem has a number of natural formulations: in particular, as the multi-dimensional "pigeon-hole principle" stated in the abstract. The history of the affine plank problem from Bang's paper to the present, together with many interesting remarks can be found in the papers [132, 133] and especially [131].

In the case of symmetric bodies, the problem is perhaps most naturally stated in terms of normed spaces. Let X be a normed space. A plank in X, is a region of the form

$$\{x \in X : \ |\phi(x) - m| \leq w\}$$

where ϕ is a functional in X^*, m a real number and w a positive number. If ϕ is taken to be a functional of norm 1, w is said to be the *half-width* of the plank. The theorem proved here is the following.

Theorem 3.7.1. *If the unit ball of a Banach space X is covered by a (countable) collection of planks in X, then the sum of the half-widths of these planks is at least 1.*

The theorem is obviously best possible in the sense that for every unit vector $\phi \in X^*$, the ball of X can be covered by one (or more) planks, perpendicular to ϕ, whose half-widths add up to 1.

The infinite-dimensional case of Theorem 1 does not follow formally from the finite-dimensional: it will be discussed and proved in Section 3 of the paper. For finite-dimensional spaces, one can restate Theorem 1, with the aid of compactness, as follows.

Theorem 3.7.2. *If $(\phi_i)_1^n$ is a sequence of unit functionals on a finite-dimensional normed space X, $(m_i)_1^n$ is a sequence of reals and $(w_i)_1^n$ a sequence of positive numbers with $\sum w_i = 1$ then there is a point x in the unit ball of X for which*

$$|\phi_i(x) - m_i| \geq w_i \quad \textit{for every} \quad i.$$

The question answered by Theorem 2 arises quite naturally in the theory of badly approximable numbers. In his paper [D], Davenport made use of the following observation. If C is a cube in $\mathbb{R}^d$ and $(H_i)_1^n$ are n hyperplanes, then there is a cube C' at least 2^{-n} times as large as C, inside C, with faces parallel to those of C, whose interior is not met by any H_i. This pigeon-hole principle can be strengthened considerably if Theorem 2 is invoked. (This was already noticed by Alexander in [A].) The estimate below immediately transfers to give sharper estimates in Davenport's theorems.

Corollary 3.7.3. *If C is a convex body, with a center of symmetry, in $\mathbb{R}^d$ and $(H_i)_1^n$ are hyperplanes, then there is a set of the form $x + \frac{1}{n+1}C$ inside C, whose interior is not met by any H_i. The result is obviously sharp for every n and C.*

Proof. Assume that C is centered at the origin and let X be the normed space represented on $\mathbb{R}^d$ with unit ball C. For each i, choose a functional ϕ_i of norm 1 in X^* and a real number m_i so that

$$H_i = \{x \in \mathbb{R}^d \colon \ \phi_i(x) = m_i\}.$$

By Theorem 2 there is a point $x \in \frac{n}{n+1}C$ with

$$|\phi_i(x) - m_i| \geq \frac{1}{n+1} \quad \text{for each} \quad i.$$

Then the set

$$x + \frac{1}{n+1}C \subset C$$

and for every y in $x + \frac{1}{n+1}C$, $\|y - x\| \leq \frac{1}{n+1}$ so that for each i, $|\phi_i(y) - \phi_i(x)| \leq \frac{1}{n+1}$: hence $\phi_i(y) - m_i$ has the same sign as $\phi_i(x) - m_i$. Thus, for each i, the whole of $x + \frac{1}{n+1}C$ lies on the same side of H_i as x does. □

Theorem 2 is readily reduced to a combinatorial theorem concerning matrices. For a sequence $(\phi_i)_1^n$ of norm 1 functionals on X, construct a matrix $A = (a_{ij})$ given by

$$a_{ij} = \phi_i(x_j), \qquad 1 \leq i, j \leq n$$

where for each j, x_j is a point in the unit ball of X at which ϕ_j attains its norm; i.e. $\phi_j(x_j) = \|x_j\| = 1$. If $(\lambda_j)_1^n$ is a sequence of reals with

$$\sum |\lambda_j| \leq 1$$

then the vector $x = \sum \lambda_j x_j$ has norm at most 1 and for each i,

$$\phi_i(x) = \sum_j a_{ij}\lambda_j.$$

Thus, Theorem 2 follows from

□

Theorem 3.7.4. *Let $A = (a_{ij})$ be an $n \times n$ matrix whose diagonal entries equal 1, $(m_i)_1^n$ a sequence of reals and $(w_i)_1^n$ a sequence of non-negative numbers with $\sum_i^n w_i \leq 1$. Then there is a sequence $(\lambda_j)_1^n$ with*

$$\sum_j |\lambda_j| \leq 1$$

and, for each i,

$$\left| \sum_j a_{ij}\lambda_j - m_i \right| \geq w_i.$$

It is also easy to see that Theorem 2′ follows immediately from Theorem 2 by regarding the rows of such a matrix as unit vectors in ℓ_∞^n. Bang effectively proved Theorem 2′ for symmetric matrices: his elegant argument is reproduced here as a lemma, since the precise statement will be needed later.

Lemma 3.7.5 (Bang)**.** *Let $H = (h_{ij})$ be a real, symmetric $n \times n$ matrix with 1's on the diagonal, $(\mu_i)_1^n$ a sequence of reals and $(\theta_i)_1^n$ a sequence of non-negative numbers. Then there is a sequence of signs $(\varepsilon_j)_1^n$ ($\varepsilon_j = \pm 1$ for each j) so that for each i,*

$$\left| \sum_j h_{ij}\varepsilon_j\theta_j - \mu_i \right| \geq \theta_i.$$

Proof. Choose signs $(\varepsilon_j)_1^n$ so as to maximise

$$\sum_{i,j} h_{ij}\varepsilon_i\varepsilon_j\theta_i\theta_j - 2\sum_i \varepsilon_i\theta_i\mu_i.$$

Fix k ($1 \leq k \leq n$) and define $(\delta_j)_1^n$ by

$$\delta_j = \{\varepsilon_j \text{if} j \neq k - \varepsilon_j \text{if} j = k.$$

Then

$$0 \leq \sum_{i,j} h_{ij}\varepsilon_i\varepsilon_j\theta_i\theta_j - 2\sum_i \varepsilon_i\theta_i\mu_i - \left(\sum_{i,j} h_{ij}\delta_i\delta_j\theta_i\theta_j - 2\sum_i \delta_i\theta_i\mu_i\right)$$

and since H is symmetric this expression is

$$4\varepsilon_k\theta_k \sum_{j\neq k} h_{kj}\varepsilon_j\theta_j - 4\varepsilon_k\theta_k\mu_k.$$

Since $h_{kk} = 1$, the latter is

$$-4\theta_k^2 + 4\varepsilon_k\theta_k \sum_j h_{kj}\varepsilon_j\theta_j - 4\varepsilon_k\theta_k\mu_k,$$

and so

$$\begin{aligned} 4\theta_k^2 &\leq 4\varepsilon_k\theta_k \left(\sum_j h_{kj}\varepsilon_j\theta_j - \mu_k\right) \\ &\leq 4\theta_k \left|\sum_j h_{kj}\varepsilon_j\theta_j - \mu_k\right|. \end{aligned}$$

Since this holds for each k, the result is proved. □

Note that the hypothesis of symmetry cannot be dropped from Lemma 3: consider, for example, the matrix $(11-11)$ for $\theta_1 = \theta_2 = 1$ and $\mu_1 = \mu_2 = 0$.

In the proof of Theorem $2'$ it may be assumed that $w_i = \frac{1}{n}$, $1 \leq i \leq n$ since planks of varying widths can be almost covered by slightly overlapping "sheets", all of the same width. (This "change of density" argument is not really needed but simplifies the succeeding arguments.) It will be shown that in this situation, Theorem $2'$ can be strengthened: the sequence $(\lambda_j)_1^n$ to be chosen will actually satisfy

$$\sum_j \lambda_j^2 \leq \frac{1}{n}$$

(which implies that $\sum_j |\lambda_j| \leq 1$ by the Cauchy-Schwartz inequality). This stronger statement can be attacked by Hilbert space methods: if the satement is true for AU where U is an orthogonal matrix, then it is true for A. Unfortunately, it is not the case that for every matrix A with 1's on the diagonal, there is an orthogonal matrix U with AU both symmetric and having large diagonal. For example, if

$$A = \left(11\frac{1}{2}1\right)$$

then the only symmetric matrices of the form AU are

$$\pm\left(111\frac{1}{2}\right) \quad \text{and} \quad \pm\left(\frac{5}{\sqrt{17}}\frac{3}{\sqrt{17}}\frac{3}{\sqrt{17}}\frac{7}{2\sqrt{17}}\right).$$

Nevertheless, Theorem $2'$ is proved by using a modified matrix A.

3.7.2 Symmetrisations of matrices and the proof of the main theorem.

The modification of a matrix, needed for the proof, is described by the following lemma. From now on, if H is a matrix, H will be said to be positive if it is symmetric and positive semi-definite.

Lemma 3.7.6. *Let A be an $n \times n$ matrix of reals, each of whose rows is non-null. Then there is a sequence $(\theta_i)_1^n$ of positive numbers and an orthogonal matrix U so that the matrix $H = (h_{ij})$ given by*

$$h_{ij} = \theta_i (AU)_{ij}$$

is positive and has 1's on the diagonal. Lemma 4 can be proved using a fixed point theorem or other topological methods. However it has an elementary proof which provides an alternative description of the sequence $(\theta_i)_1^n$. Recall that for a matrix B, the trace-class, or nuclear, norm $\|B\|_{C_1}$ of B, is $tr(H)$, where H is the positive square root of BB^*. By the Cauchy-Schwartz inequality

$$\|B\|_{C_1} = \max\{tr(BU) \colon \ U \text{ orthogonal}\}.$$

Also by the Cauchy-Schwartz inequality, if B and C are $n \times n$ matrices then

$$\|BC\|_{C_1} \leq (tr(BB^*))^{1/2} (tr(CC^*))^{1/2}.$$

Before the proof of Lemma 4 it will be convenient to prove the lemma that really forms the crux of the proof of Theorem 2. The estimate is somewhat unusual since it involves the sum of the squares of the diagonal entries of a matrix: nevertheless, it is a consequence of the matrix Cauchy-Schwartz inequality.

Lemma 3.7.7. *If $H = (h_{ij})$ is a positive matrix with non-zero diagonal entries and U is orthogonal then*

$$\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \leq \sum_i h_{ii}.$$

Proof. For each i let $\gamma_i = \frac{(HU)_{ii}}{h_{ii}}$ and let D be the diagonal matrix, $diag(\gamma_i)_1^n$ and T, the positive square root of H. Then

$$\begin{aligned}\sum_i \frac{(HU)_{ii}^2}{h_{ii}} &= \sum_i \gamma_i (HU)_{ii} \\ &= tr\ DHU \le \|DH\|_{C_1} \\ &= \|(DT)T\|_{C_1} \\ &\le [tr\ DT(DT)^*]^{\frac{1}{2}} [tr\ TT^*]^{\frac{1}{2}} \\ &= [tr\ DHD]^{\frac{1}{2}} [tr\ H]^{\frac{1}{2}} \\ &= \left(\sum_i \gamma_i^2 h_{ii}\right)^{\frac{1}{2}} \left(\sum_i h_{ii}\right)^{\frac{1}{2}} \\ &= \left(\sum \frac{(HU)_{ii}^2}{h_{ii}}\right)^{\frac{1}{2}} \left(\sum h_{ii}\right)^{\frac{1}{2}}.\end{aligned}$$

Hence

$$\left(\sum_i \frac{(HU)_{ii}^2}{h_{ii}}\right)^{\frac{1}{2}} \le \left(\sum_i h_{ii}\right)^{\frac{1}{2}}.$$

□

□

Lemma 5 immediately implies:

Lemma 3.7.8. *If $H = (h_{ij})$ is a positive $n \times n$ matrix with non-zero diagonal entries, then*

$$\left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{ij} \right) \right\|_{C_1} \le \sqrt{n} \|H\|_{C_1}^{1/2}.$$

Proof. There is some orthogonal matrix U for which

$$\left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{ij} \right) \right\|_{C_1} = \sum_i \frac{1}{\sqrt{h_{ii}}} (HU)_{ii}.$$

By the Cauchy-Schwartz inequality this is at most

$$\sqrt{n} \left(\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \right)^{\frac{1}{2}}$$

and this is at most $\sqrt{n}\big(\sum h_{ii}\big)^{\frac{1}{2}} = \sqrt{n}\|H\|_{C_1}$ by Lemma 5.

□

Proof of Lemma 4. Plainly it suffices to find $(\theta_i)_1^n$ positive and U orthogonal so that $(\theta_i (AU)_{ij})$ is positive and has constant, non-zero, diagonal.

Since the rows of A are non-null, there is a constant $c > 0$ so that if $(\theta_i)_1^n$ is a sequence of positive numbers

$$\|(\theta_i a_{ij})\|_{C_1} \geq c \max_i \theta_i.$$

Since $\|(\theta_i a_{ij})\|_{C_1}$ is continuous as a function of $(\theta_i)_1^n$, there is a sequence $(\theta_i)_1^n$ of positive numbers which minimises

$$\|(\theta_i a_{ij})\|_{C_1}$$

subject to the condition $\prod_i \theta_i = 1$. Let $H = (h_{ij})$ be the positive square root of $(\theta_i (AA^*)_{ij} \theta_j)$, for this particular sequence, and note that there is an orthogonal matrix U for which

$$h_{ij} = \theta_i (AU)_{ij}, \quad 1 \leq i, j \leq n.$$

Again, since A has non-null rows, the diagonal entries of H are non-zero. For each i, let

$$\gamma_i = \frac{1}{\sqrt{h_{ii}}} \left(\prod_{j=1}^n \sqrt{h_{jj}} \right)^{1/n}.$$

Since $\prod_i^n \gamma_i = 1$, the matrix $(\gamma_i \theta_i a_{ij})$ has nuclear norm at least that of $(\theta_i a_{ij})$, the latter being $\|H\|_{C_1}$. So

$$\begin{aligned}
\|H\|_{C_1} &\leq \|(\gamma_i \theta_i a_{ij})\|_{C_1} \\
&= \|(\gamma_i h_{ij})\|_{C_1} \\
&= \left(\prod_{k=1}^n \sqrt{h_{kk}} \right)^{\frac{1}{n}} \left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{ij} \right) \right\|_{C_1} \\
&\leq \sqrt{n} \|H\|_{C_1}^{\frac{1}{2}} \left(\prod_k \sqrt{h_{kk}} \right)^{\frac{1}{n}}
\end{aligned}$$

by Lemma 6. So

$$\left(\frac{1}{n} \sum_i h_{ii} \right)^{\frac{1}{2}} \leq \left(\prod_k h_{kk} \right)^{\frac{1}{2n}}$$

implying that the h_{ii}'s are all the same. □

□

Proof of Theorem 3. The statement to be proved is that if $A = (a_{ij})$ is a real $n \times n$ matrix with 1's on the diagonal and $(m_i)_1^n$ is a sequence of reals, then there is a sequence $(\lambda_j)_1^n$ of reals with

$$\sum_j \lambda_j^2 \le \frac{1}{n}$$

and, for every i

$$\left|\sum_j a_{ij}\lambda_j - m_i\right| \ge \frac{1}{n}.$$

Using Lemma 4, choose a sequence $(\theta_j)_1^n$ of positive numbers and an orthogonal matrix U, so that if

$$H = (\theta_i (AU)_{ij}), \tag{2}$$

then H is positive and has 1's on the diagonal.

By Lemma 3, there is a choice of signs $(\varepsilon_j)_1^n$ so that for each i,

$$\left|\sum_j h_{ij}\varepsilon_j\theta_j - n\theta_i m_i\right| \ge \theta_i. \tag{3}$$

From (2) and (3), one has that for each i,

$$\left|\theta_i \sum_j (AU)_{ij}\varepsilon_j\theta_j - n\theta_i m_i\right| \ge \theta_i,$$

and hence

$$\left|\sum_k a_{ik}\left(\frac{1}{n}\sum_j u_{kj}\varepsilon_j\theta_j\right) - m_i\right| \ge \frac{1}{n}.$$

For each k set

$$\lambda_k = \frac{1}{n}\sum_j u_{kj}\varepsilon_j\theta_j.$$

It remains to check that $\sum \lambda_k^2 \le \frac{1}{n}$. But $\sum \lambda_k^2 = \frac{1}{n^2}\sum \theta_j^2$ since U is orthogonal and so what is needed is

$$\sum_j \theta_j^2 \le n.$$

From (2),

$$\theta_i a_{ij} = (HU^*)_{ij} \quad \text{for all} \quad i \text{ and } j$$

and so in particular, since $a_{ii} = 1$ for each i,

$$\theta_i = (HU^*)_{ii}, \quad 1 \leq i \leq n.$$

Now since $h_{ii} = 1$ for each i, Lemma 5 shows that

$$\begin{aligned}\sum_i \theta_i^2 = \sum_i (HU^*)_{ii}^2 &= \sum_i \frac{(HU^*)_{ii}^2}{h_{ii}} \\ &\leq \sum_i h_{ii} = n. \qquad \square\end{aligned}$$

□

3.7.3 The infinite-dimensional case.

Theorem 2 and weak*-compactness immediately imply the following "multiple Hahn-Banach" theorem.

Theorem 3.7.9. *Let $(x_i)_1^\infty$ be a sequence of unit vectors in a normed space X, $(m_i)_1^\infty$ a sequence of real numbers and $(w_i)_1^\infty$ a sequence of non-negative reals with $\sum_1^\infty w_i \leq 1$. Then there is a linear functional ϕ of norm at most 1 in X^* with*

$$|\phi(x_i) - m_i| \geq w_i \quad \textit{for every} \quad i.$$

For reflexive spaces, Theorem 1 follows immediately from Theorem 2. For general spaces, Theorem 1 is a little more delicate. It can be regarded as a quantitative strengthening of the Banach-Steinhaus theorem. If Φ is an unbounded subset of the dual X^* of a Banach space X, then there are elements $\phi_1, \phi_2, \ldots$ of Φ with (say)

$$\sum_1^\infty n\|\phi_n\|^{-1} < 1.$$

By Theorem 1, there is a point $x \in X$ of norm at most 1 so that for each n,

$$\left| \frac{\phi_n}{\|\phi_n\|}(x) \right| > n\|\phi_n\|^{-1},$$

i.e. $|\phi_n(x)| > n$.

To prove Theorem 1 it is necessary to examine the proof of Theorem 2 more closely. The change of density argument in Section 1 and the proof in Section 2 actually yield the following strong form of Theorem $2'$. Theorem 1 will be deduced from this.

Theorem 3.7.10. *Let* (a_{ij}) *be a real,* $n \times n$ *matrix with 1's on the diagonal,* $(m_i)_1^n$ *a sequence of real numbers and* $(w_i)_1^n$*, a sequence of positive numbers. Then there is a sequence* $(\lambda_j)_1^n$ *with*

$$\sum_j w_j^{-1}\lambda_j^2 \le \sum_j w_j$$

and, for every i*,*

$$\left|\sum_j a_{ij}\lambda_j - m_i\right| \ge w_i. \qquad \square$$

Proof of Theorem 1. Suppose $(\phi_i)_1^\infty$ are unit functionals in $X^*, (m_i)_1^\infty$ are real numbers and $(w_i)_1^\infty$ are non-negative numbers with $\sum\limits_i w_i < 1$. The problem is to find a point x in the unit ball of X with

$$|\phi_i(x) - m_i| > w_i \quad \text{for each} \quad i.$$

Choose a sequence $(v_i)_1^\infty$ with

$$v_i > w_i \ge 0 \quad \text{for each} \quad i$$

but

$$\sum_i v_i = 1 - \varepsilon < 1.$$

For each i, choose a point x_i of norm at most 1 in X with $\phi_i(x_i) = 1-\varepsilon$. Applying Theorem 8 to the first n functionals and vectors one obtains, for each n, a sequence $(\lambda_j^{(n)})_{j=1}^n$ satisfying

$$\sum_{j=i}^n v_i^{-1}(\lambda_j^{(n)})^2 \le (1-\varepsilon)^{-2}\sum_{j=i}^n v_j < (1-\varepsilon)^{-1} \tag{5}$$

and for $1 \le i \le n$

$$\left|\phi_i\left(\sum_{j=i}^n \lambda_j^{(n)} x_j\right) - m_i\right| \ge v_i.$$

Regard $(\lambda_j^{(n)})$ as an infinite sequence by filling out with zeroes. From (5), for each n,

$$\sum_{j=1}^\infty |\lambda_j^{(n)}| \le \left(\sum_{j=1}^\infty v_j\right)^{1/2}\left(\sum_{j=1}^\infty v_j^{-1}(\lambda_j^{(n)})^2\right)^{1/2} \tag{6}$$
$$< (1-\varepsilon)^{1/2}(1-\varepsilon)^{-1/2} = 1.$$

Moreover, for each m and n,

$$\sum_{j=m}^{\infty} |\lambda_j^{(n)}| \leq \left(\sum_{j=m}^{\infty} v_j \right)^{1/2} (1-\varepsilon)^{-1/2}.$$

Since the right-hand side $\to 0$ as $m \to \infty$, the sequences $(\lambda_j^{(n)})_{j=1}^{\infty}$ are uniformly summable, so the collection has a (norm) limit point $(\lambda_j)_1^{\infty}$ (say) in ℓ_1. From (6), the point $x = \sum_j \lambda_j x_j \in X$ has norm at most 1 and clearly

$$|\phi_i(x) - m_i| \geq v_i > w_i \quad \text{for every} \quad i.$$

□

3.8 On quotients of Banach spaces having shrinking unconditional bases

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Abstract:
It is proved that if a Banach space Y is a quotient of a Banach space having a shrinking unconditional basis, then every normalized weakly null sequence in Y has an unconditional subsequence. The proof yields the corollary that every quotient of Schreier's space is c_o-saturated.

3.8.1 Introduction.

We shall say that a Banach space Y has *property (WU)* if every normalized weakly null sequence in Y has an unconditional subsequence. The well known example of Maurey and Rosenthal [134] shows that not every Banach space has property (WU) (see also [135]). W.B. Johnson [136] proved that if Y is a quotient of a Banach space X having a shrinking unconditional f.d.d. and the quotient map does not fix a copy of c_0, then Y has (WU). Our main result extends this (and solves Problem IV.1 of [136]).

Theorem 3.8.1 (A)**.** *Let X be a Banach space having a shrinking unconditional finite dimensional decomposition. Then every quotient of X has property (WU).*

Of course such an X will itself have property (WU). Furthermore, if (E_n) is an unconditional f.d.d. (finite dimensional decomposition) for X, then (E_n) is shrinking if and only if X does not contain ℓ_1. The proof of Theorem A yields

Theorem 3.8.2 (B)**.** *Let Y be a Banach space which is a quotient of S, the Schreier space. Then Y is c_o-saturated. Y is said to be c_o-saturated if every infinite dimensional subspace of Y contains an isomorph of c_0.*

**

Our notation is standard as may be found in the books of Lindenstrauss and Tzafriri [137, 138]. The proof of Theorem A is given in §1 and the proof of Theorem B appears in §2. §3 contains some open problems. We thank H. Rosenthal and T. Schlumprecht for useful conversations regarding the results contained herein.

3.8.2 The proof of Theorem A.

Let T be a bounded linear operator from X onto Y where X has a shrinking unconditional f.d.d., $(\approx{E}_i)$. By renorming if necessary we may suppose that $(\approx{E}_i)$ is 1-unconditional. Y^* is separable and so by a theorem of Zippin [139] we may assume that Y is a subspace of a Banach space Z possessing a bimonotone shrinking basis, (z_i). Fix $C > 0$ such that

$$T(CB_aX) \supseteq B_aY \equiv \{y \in Y : \|y\| \leq 1\}\ .$$

Recall that $(\widetilde{E_i})$ is a *blocking* of $(\approx{E}_i)$ if there exist integers $0 = q_0 < q_1 < q_2 < \cdots$ such that $\widetilde{E_i} = [\approx{E}_j]_{j=q_{i-1}+1}^{q_i}$ for all i (where $[\cdots]$ denotes the closed linear span). Similarly, $\widetilde{F_i} = [z_j]_{j=q_{i-1}+1}^{q_i}$ defines a blocking of (z_i).

Fix a sequence $\varepsilon_{-1} > \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$ converging to 0 which satisfies

$$\sum_{i=-1}^{\infty} \varepsilon_i < 1/4 \quad \text{and} \quad \sum_{i=p}^{\infty} (4i+2)\varepsilon_i < \varepsilon_{p-1} \quad \text{for} \quad p \geq 0\ . \tag{1.1}$$

Then choose $\tilde{\varepsilon}_0 > \tilde{\varepsilon}_1 > \cdots$ converging to 0 which satisfies

$$4p\tilde{\varepsilon}_p < \varepsilon_{p+2} \quad \text{for} \quad p \geq 1 \quad \text{and} \quad \sum_{j=p+1}^{\infty} \tilde{\varepsilon}_j < \tilde{\varepsilon}_p \quad \text{for} \quad p \geq 0\ . \tag{1.2}$$

Our first step is the blocking technique of Johnson and Zippin.

Lemma 3.8.3. *[140, 141] There exist blockings $(\widetilde{E_i})$ and $(\widetilde{F_i})$ of $(\approx{E}_i)$ and (z_i), respectively, such that if $(\widetilde{Q_i})$ is the sequence of finite rank projections on Z associated with $(\widetilde{F_i})$ then*

(1.3) *For all $i \in \mathbb{N}$ and $x \in \widetilde{E_i}$ with $\|x\| \leq C$, we have $\|\widetilde{Q_j}Tx\| < \tilde{\varepsilon}_{\max(i,j)}$ if $j \neq i, i-1$.*

Roughly, this says that $T\widetilde{E_i}$ is essentially contained in $\widetilde{F_{i-1}} + \widetilde{F_i}$ (where $\widetilde{F_0} = \{0\}$). Let (y_i'') be a normalized weakly null sequence in Y. Choose a subsequence (y_i'') of (y_i) and a blocking (F_i) of $(\widetilde{F_i})$, given by $F_i = [\widetilde{F_j}]_{j=q_{i-1}+1}^{q_i}$, such that if $Q_i = \sum_{j=q_{i-1}+1}^{q_i} \widetilde{Q_j}$ is the sequence of finite rank projections on Z associated with (F_i), then

$$\|Q_j y_i'\| < \tilde{\varepsilon}_{\max(i,j)} \text{ if } i \neq j\ . \tag{1.4}$$

Roughly, y_i' is essentially in F_i. Furthermore we may assume that

$$\|\sum a_i y_i'\| = 1 \text{ implies } \max |a_i| \leq 2\ . \tag{1.5}$$

Let (E_i) be the blocking of $(\widetilde{E_i})$ given by the *same* sequence (q_i) which defined (F_i), $E_i = [\widetilde{E_j}]_{j=q_{i-1}+1}^{q_i}$.

We begin with a sequence of elementary technical yet necessary lemmas.

For $I \subseteq \mathbb{N}$ we define $Q_I = \sum_{j\in I} Q_j$ and set $Q_\phi = 0$.

Lemma 3.8.4. *Let* $0 \leq n < m$ *be integers and let* $y = \sum_{i\notin(n,m)} a_i y'_i$ *with* $\|y\| = 1$. *Then for* $j \in (n,m)$, $\|Q_j y\| < \varepsilon_j$ *and* $\|Q_{(n,m)}y\| < \varepsilon_n$.

Proof. Let $n < j < m$. Then by (1.5), (1.4), (1.2) and (1.3),

$$\begin{aligned}\|Q_j y\| &\leq 2\Big(\sum_{i\leq n}\|Q_j y'_i\| + \sum_{i\geq m}\|Q_j y'_i\|\Big)\\ &< 2(n\tilde{\varepsilon}_j + \tilde{\varepsilon}_{m-1})\\ &\leq (2j+2)\tilde{\varepsilon}_j \leq 4j\tilde{\varepsilon}_j < \varepsilon_j\end{aligned}$$

Thus $\|Q_{(n,m)}y\| < \sum_{j\in(n,m)} \varepsilon_j < \varepsilon_n$ by (1.1). ☐

Lemma 3.8.5. *Let* $0 = p_0 < r_0 = 1 < p_1 < r_1 < p_2 < r_2 < \cdots$ *be integers and let* $y = \sum_{i=1}^{\infty} a_i y'_{p_i}$ *with* $\|y\| = 1$. *Then for* $i \in \mathbb{N}$,

$$\|Q_{[r_{i-1},r_i)}y - a_i y'_{p_i}\| < \varepsilon_{p_{i-1}-1}\ .$$

Proof.

$$\begin{aligned}&\|Q_{[r_{i-1},r_i)}y - a_i y'_{p_i}\|\\ &\leq \|Q_{[r_{i-1},r_i)}\sum_{j\neq i} a_j y'_{p_j}\| + \|Q_{[r_{i-1},r_i)}a_i y'_{p_i} - a_i y'_{p_i}\|\\ &\text{which by lemma 1.2 is}\\ &< \varepsilon_{r_{i-1}-1} + \|Q_{[1,r_{i-1})}a_i y'_{p_i}\| + \|Q_{[r_i,\infty)}a_i y'_{p_i}\|\\ &< \varepsilon_{r_{i-1}-1} + 2\sum_{k<r_{i-1}}\|Q_k y'_{p_i}\| + 2\varepsilon_{r_i-1} \text{ (by (1.5) and lemma 1.2)}\\ &< \varepsilon_{r_{i-1}-1} + 2(r_{i-1}-1)\tilde{\varepsilon}_{p_i} + 2\varepsilon_{r_i-1} \text{ (by (1.4))}\\ &\leq \varepsilon_{p_{i-1}} + 2p_i\tilde{\varepsilon}_{p_i} + 2\varepsilon_{p_i} < \varepsilon_{p_{i-1}} + 4\varepsilon_{p_i} \text{ (by (1.2))}\\ &< \varepsilon_{p_{i-1}-1} \text{ (by 1.1)}\ .\end{aligned}$$

☐

Lemma 3.8.6. *Let* $i \in \mathbb{N}$, $x \in E_i$ *and* $\|x\| \leq C$. *Then*

$$\|Q_j Tx\| < \varepsilon_{\max(i,j)} \text{ if } j \neq i, i-1\ ,$$
$$\|Q_{[1,i-2]}Tx\| < \varepsilon_{i-1} \text{ and } \|Q_{[i,\infty)}Tx\| < \varepsilon_{i-1}\ .$$

Proof. Let $x = \sum_{\ell\in(q_{i-1},q_i]} \omega_\ell$ with $\omega_\ell \in \widetilde{E_\ell}$.

$$\begin{aligned}\|Q_j Tx\| &\leq \sum_{k\in(q_{j-1},q_j]}\sum_{\ell\in(q_{i-1},q_i]}\|\widetilde{Q_k}T\omega_\ell\|\\ (\text{if } j < i-1) : &< \sum_{k\in(q_{j-1},q_j]}\sum_{\ell\in(q_{i-1},q_i]}\tilde{\varepsilon}_\ell \text{ (by (1.4))}\end{aligned}$$

$$
\begin{aligned}
&< q_j\tilde{\varepsilon}_{q_{i-1}} < q_{i-1}\tilde{\varepsilon}_{q_{i-1}} < \varepsilon_{q_{i-1}+2} < \varepsilon_i \text{ using (1.2)} \\
(\text{if } j > i) : &< \sum_{k\in(q_{j-1},q_j]} \sum_{\ell\in(q_{i-1},q_i]} \tilde{\varepsilon}_k \\
&< \sum_{k\in(q_{j-1},q_j]} q_i\tilde{\varepsilon}_k \le q_i\tilde{\varepsilon}_{q_{j-1}} \\
&\le q_{j-1}\tilde{\varepsilon}_{q_{j-1}} < \varepsilon_{q_{j-1}+2} \le \varepsilon_{j+1} < \varepsilon_j .
\end{aligned}
$$

Finally,

$$\|Q_{[1,i-2]}Tx\| \le \sum_{k=1}^{i-2} \|Q_kTx\| < \sum_{k=1}^{i-2} \varepsilon_i = (i-2)\varepsilon_i < \varepsilon_{i-1}$$

and

$$\|Q_{[i,\infty)}Tx\| \le \sum_{k=i}^{\infty} \|Q_kTx\| < \sum_{k=i}^{\infty} \varepsilon_k < \varepsilon_{i-1} \ .$$

□

Lemma 3.8.7. *Let* $\|x\| \le C, x = \sum_{k\neq j,j+1} \omega_k$ *where* $\omega_k \in E_k$ *for all* k*. Then*

$$\|Q_jTx\| < \varepsilon_{j-1}$$

Proof. By lemma 1.4,

$$
\begin{aligned}
\|Q_jTx\| &\le \sum_{k\neq j,j+1} \|Q_jT\omega_k\| < \sum_{k<j} \varepsilon_j + \sum_{k>j+1} \varepsilon_k \\
&< (j-1)\varepsilon_j + \varepsilon_j = j\varepsilon_j < \varepsilon_{j-1}
\end{aligned}
$$

□

Lemma 3.8.8. *Let* $1 \le n < m$ *and* $x = \sum \omega_j, \|x\| \le C$*, with* $\omega_j \in E_j$ *for all* j*. Suppose that* $\|Q_jTx\| < 2\varepsilon_{j-1}$ *for* $n < j < m$*. Let* $a_{j-1} = Q_{j-1}T\omega_j$ *and* $b_j = Q_jT\omega_j$*. Then*
a) $\|a_j + b_j\| < 3\varepsilon_{j-1}$ *for* $n < j < m$ *and*
b) $\left\|\sum_{j\in(r,s]} T\omega_j - (a_r + b_s)\right\| < 5\varepsilon_{r-1}$ *if* $n < r < s < m$*.*

Proof. a) Let $n < j < m$. By lemma 1.5, $\|Q_jTx - (a_j + b_j)\| = \left\|Q_j\left(\sum_{i\neq j,j+1} T\omega_i\right)\right\| < \varepsilon_{j-1}$. Since $\|Q_jTx\| < 2\varepsilon_{j-1}$, a) follows.
b) Let $n < r < s < m$ and let $j \in (r,s]$. Then $T\omega_j = a_{j-1} + b_j + \gamma_j$ where $\|\gamma_j\| < 2\varepsilon_{j-1}$ by lemma 1.4. Thus

$$
\begin{aligned}
&\left\| \sum_{r+1}^{s} T\omega_j - (a_r + b_s) \right\| \\
&\leq \| a_r + b_{r+1} + a_{r+1} + b_{r+2} + \cdots + a_{s-1} + b_s - (a_r + b_s) \| + \sum_{j=r+1}^{s} 2\varepsilon_{j-1} \\
&< \sum_{r+1}^{s-1} \| a_j + b_j \| + 2\varepsilon_{r-1} \\
&< \sum_{r+1}^{s-1} 3\varepsilon_{j-1} + 2\varepsilon_{r-1} \text{ (by a)} \Bigg) \\
&< 5\varepsilon_{r-1}
\end{aligned}
$$

We next come to the key lemma. Let (P_j) be the sequence of finite rank projections on X associated with (E_j). For $I \subseteq \mathbb{N}$, we let $P_I = \sum_{i \in I} P_i$.

$\square$

notation: If $x = \sum x_j \in X$ with $x_j \in E_j$ for all j and $\bar{x} \in X$, we define

$$\bar{x} \precsim x \text{ if } \bar{x} = \sum a_j x_j \text{ with } 0 \leq a_j \leq 1 \text{ for all } j$$

Lemma 3.8.9. *Let $n \in \mathbb{N}$ and let $\varepsilon > 0$. There exists $m \in \mathbb{N}, m > n+1$, such that whenever $x \in CBaX$ with $\|Q_j T x\| < 2\varepsilon_{j-1}$ for all $j \in (n, m)$ then: there exists $\bar{x} \precsim x$ with*

1. *$\|Tx - T\bar{x}\| < \varepsilon$ and*
2. *$P_r \bar{x} = 0$ for some $r \in (n, m)$.*

Remark 34. *Lemma 1.7 is the main difference between our result and Johnson's earlier special case* [J]. *In the case where T does not fix a copy of c_0, Johnson showed that one could take $\bar{x} = x - P_r(x)$ for some $r \in (n, m)$.*

The proof of lemma 1.7 requires the following key

Sublemma 1.8. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. There exists an integer $m = m(n, \varepsilon) > n+1$ satisfying the following. Let $x \in CBaX, x = \sum \omega_j$ with $\omega_j \in E_j$ for all j. Assume in addition that $\|Q_j T x\| < 2\varepsilon_{j-1}$ for $j \in (n, m)$ and set $a_{j-1} = Q_{j-1} T \omega_j$ and $b_j = Q_j T \omega_j$. Then there exist $k \in \mathbb{N}$ and integers $n < i_1 < \cdots < i_k < m$ such that

$$k^{-1} \left\| a_{i_1} + a_{i_2} + \cdots + a_{i_k} \right\| < \varepsilon \tag{1.6}$$

Proof of Lemma 1.7. Let $n \in \mathbb{N}$ and $\varepsilon > 0$. Choose $n_0 \geq n$ such that

$$\varepsilon_{n_0} < \varepsilon / 12 \tag{1.7}$$

Let $m_1 = m\left(n_0 + 1, \varepsilon/3\right)$ be given by the sublemma and let $m = m\left(m_1, \varepsilon/3\right)$.

Let $x = \sum \omega_j \in CBaX$ with $\omega_j \in E_j$ for all j and suppose that $\|Q_j Tx\| < 2\varepsilon_{j-1}$, $a_{j-1} = Q_{j-1}T\omega_j$ and $b_j = Q_j T\omega_j$ for $j \in (n, m)$. By our choice of m there exist integers k and K and integers $n \le n_0 < n_0 + 1 < i_1 < i_2 < \cdots < i_k < m_1 < j_1 < \cdots < j_K < m$ such that

$$k^{-1}\|a_{i_1} + \cdots + a_{i_k}\| < \varepsilon/3 \text{ and} \tag{1.8}$$

$$K^{-1}\|a_{j_1} + \cdots + a_{j_K}\| < \varepsilon/3 \tag{1.9}$$

Define

$$\begin{aligned}\bar{x} = &\sum_{1}^{i_1}\omega_j + \frac{k-1}{k}\sum_{i_1+1}^{i_2}\omega_j + \cdots + \frac{1}{k}\sum_{i_{k-1}+1}^{i_k} + \frac{0}{k}\sum_{i_k+1}^{j_1}\omega_j \\ &+ \frac{1}{K}\sum_{j_1+1}^{j_2}\omega_j + \cdots + \frac{K}{K}\sum_{j_k+1}^{\infty}\omega_j\end{aligned}$$

Clearly (2) holds and we are left to check (1).

$$\begin{aligned}\|Tx - T\bar{x}\| = \|&\frac{1}{k}\sum_{i_1+1}^{i_2}T\omega_j + \frac{2}{k}\sum_{i_2+1}^{i_3}T\omega_j \\ &+ \cdots + \frac{k}{k}\sum_{i_k+1}^{j_1}T\omega_j + \frac{K-1}{K}\sum_{j_1+1}^{j_2}T\omega_j + \cdots + \frac{1}{K}\sum_{j_{K-1}+1}^{j_K}T\omega_j\|\end{aligned}$$

Thus by lemma 1.6 ,

$$\begin{aligned}\|Tx - T\bar{x}\| \le \|&\frac{1}{k}a_{i_1} + \frac{1}{k}b_{i_2} + \frac{2}{k}a_{i_2} + \frac{2}{k}b_{i_2} \\ &+ \cdots + \frac{k}{k}a_{i_k} + \frac{K}{K}b_{j_1} + \frac{K-1}{K}a_{j_1} + \frac{K-1}{K}b_{j_2} + \cdots + \frac{1}{K}a_{j_{K-1}} + \frac{1}{K}b_{j_K}\| \\ &+ k^{-1}\sum_{j=1}^{k}5j\varepsilon_{i_{j-1}} + K^{-1}\sum_{\ell=1}^{K}5\ell\varepsilon_{j_\ell-1}\end{aligned}$$

Now $k^{-1}\sum_{j=1}^{k}5j\varepsilon_{i_j-1} \le 5\sum_{j=1}^{k}\varepsilon_{i_j-1} < \varepsilon_{i_1-2} \le \varepsilon_{n_0}$ and $K^{-1}\sum_{\ell=1}^{K}5\ell\varepsilon_{j_\ell-1} < \varepsilon_{n_0}$ as well. Thus

$$\begin{aligned}\|Tx - T\bar{x}\| < &k^{-1}\|a_{i_1} + \cdots + a_{i_k}\| + K^{-1}\|b_{j_1} + \cdots + b_{j_K}\| \\ &+ k^{-1}\sum_{j=2}^{k}\|b_{i_j} + a_{i_j}\| + K^{-1}\sum_{\ell=1}^{K-1}\|b_{j_\ell} + a_{j_\ell}\| + 2\varepsilon_{n_0}\end{aligned}$$

From (1.8), (1.9) and lemma 1.6 we obtain

$$
\begin{aligned}
\|Tx - T\bar{x}\| &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \sum_{j=2}^{k} 3\varepsilon_{i_j-1} + \sum_{\ell=1}^{K-1} 3\varepsilon_{j_\ell-1} + 2\varepsilon_{n_0} \\
&< \frac{2\varepsilon}{3} + \varepsilon_{n_0} + \varepsilon_{n_0} + 2\varepsilon_{n_0} < \varepsilon
\end{aligned}
$$

(by (1.7)). ☐

Proof of Sublemma 1.8. If the sublemma fails then by a standard compactness argument we obtain $\omega_j \in E_j$ for $j \in \mathbb{N}$ such that for all m,

$$\left\|\sum_{j=1}^{m} \omega_j\right\| \le C \text{ and } \left\|Q_j T\left(\sum_{i=1}^{m} \omega_i\right)\right\| \le 3\varepsilon_{j-1}$$

if $n < j < m$. The extra ε_{j-1} comes from an application of lemma 1.5. Furthermore setting $Q_{j-1}T\omega_j = a_j$ and $Q_j T\omega_j = b_j$ for $j \in \mathbb{N}$, then for all k and all $n < i_1 < \cdots < i_k$ we have

$$k^{-1}\left\|a_{i_1} + \cdots + a_{i_k}\right\| \ge \varepsilon \tag{1.10}$$

Now $a_j \in F_j$ and (F_j) is a shrinking f.d.d. Thus $(a_j)_{j>n}$ is a seminormalized weakly null sequence. By (1.10) any spreading model of a subsequence of (a_j) must be equivalent to the unit vector basis of ℓ_1 (see [BL] for basic information on spreading models). In particular we can choose an even integer k and integers $n < i_1 < \cdots < i_k$ such that

$$\left\|a_{i_1} - a_{i_2} + \cdots + a_{i_{k-1}} - a_{i_k}\right\| > C\|T\| + 1 \tag{1.11}$$

However,

$$
\begin{aligned}
C\|T\| \ge{}& \left\|T\left(\sum_{i_1+1}^{i_2} \omega_j + \sum_{i_3+1}^{i_4} + \cdots + \sum_{i_{k-1}+1}^{i_k} \omega_j\right)\right\| \\
\ge{}& \left\|a_{i_1} + b_{i_2} + a_{i_3} + b_{i_4} + \cdots + a_{i_{k-1}} + b_{i_k}\right\| \\
&- 5\sum_{j=1}^{k} \varepsilon_{i_j-1} (\text{ by lemma 1.6})
\end{aligned}
$$

Now $5\sum_{j=1}^{k} \varepsilon_{i_j-1} < \varepsilon_{i_1-2}$ and by lemma 1.6 and (1.11)

$$
\begin{aligned}
&\left\|a_{i_1} + b_{i_2} + \cdots + a_{i_{k-1}} + b_{i_k}\right\| \\
&\quad \ge \left\|a_{i_1} - a_{i_2} + a_{i_3} - a_{i_4} + \cdots + a_{i_{k-1}} - a_{i_k}\right\| \\
&\quad - \sum_{j=1}^{k/2} \left\|a_{i_{2j}} + b_{i_{2j}}\right\| > C\|T\| + 1 - \sum_{j=1}^{k/2} 3\varepsilon_{i_{2_j}-1}
\end{aligned}
$$

Thus

$$\begin{aligned} C\|T\| &> C\|T\| + 1 - \varepsilon_{i_1-2} - \varepsilon_{i_2-2} \\ &\geq C\|T\| + 1 - 2\varepsilon_{i_1-2} > C\|T\| \end{aligned}$$

which is impossible.

Completion of the proof of Theorem A.

Let the integer m given by lemma 1.7 be denoted by $m = m(n;\varepsilon)$. Choose $1 < p_1 < p_2 < \cdots$ such that for all $i, p_{i+1} - 1 \geq m\left(p_i; \varepsilon_{p_i}\right)$. Let $(y_i) = \left(y'_{p_i}\right)$. We shall prove that (y_i) is unconditional.

Let $y = \sum a_i y_i, \|y\| = 1, x \in CBaX, Tx = y$ and let $x = \sum_{i=0}^{\infty} g_i$ where $g_0 = P_{[1,p_1)}x$ and $g_i = P_{[p_i,p_{i+1})}x$ for $i \geq 1$. We shall apply lemma 1.7 to each g_i for $i \geq 1$. Fix $i \geq 1$ and let $(n,m) = (p_i, p_{i+1} - 1)$. Let $j \in (n,m)$. Then $\|Q_j y\| < \varepsilon_j$ by lemma 1.2. Thus $\|Q_j Tx\| = \left\|Q_j T g_i + Q_j T \sum_{k \neq i} g_k\right\| < \varepsilon_j$. However $\left\|Q_j T \sum_{k \neq i} g_k\right\| < \varepsilon_{j-1}$ by lemma 1.5 so $\|Q_j T g_i\| < \varepsilon_{j-1} + \varepsilon_j < 2\varepsilon_{j-1}$. Thus by lemma 1.7 there exist $\bar{g}_i \precsim g_i$ and $r_i \in (p_i, p_{i+1} - 1)$ such that $P_{r_i}\bar{g}_i = 0$ and $\|Tg_i - T\bar{g}_i\| < \varepsilon_{p_i}$ for all $i \in \mathbb{N}$.

Let $\bar{x} = \sum_{i=0}^{\infty} \bar{g}_i = \sum_{i=1}^{\infty} \bar{x}_i$ where $\bar{g}_0 = g_0$ and $\bar{x}_i = p_{[r_{i-1},r_i)}\bar{x}$ for $i \in \mathbb{N}\,(r_0 = 1)$. Of course, $\bar{x}_i = P_{(r_{i-1},r_i)}\bar{x}$ if $i > 1$.

Claim: $\|T\bar{x}_i - a_i y_i\| < 4\varepsilon_{p_{i-1}-1}$ for $i \in \mathbb{N}$.

Indeed $\left\|Q_{[r_{i-1},r_i)}y - a_i y_i\right\| < \varepsilon_{p_{i-1}-1}$ by lemma 1.3. Thus the claim follows from the Subclaim: $\left\|Q_{[r_{i-1},r_i)}Tx - T\bar{x}_i\right\| < 3\varepsilon_{p_{i-1}-1}$.

To see this we first note that

$$\begin{aligned} &\|Q_{[r_{i-1},r_i)}Tx - Q_{[r_{i-1},r_i)}T\left(g_{i-1} + g_i + g_{i+1}\right)\| \\ &\qquad \leq \sum_{k \in [r_{i-1},r_i)} \left\|Q_k \sum_{j \neq i-1,i,i+1} Tg_j\right\| \\ &\qquad < \sum_{k \in [r_{i-1},r_i)} \varepsilon_{k-1} \quad (\text{ by lemma 1.5}) \\ &\qquad < \varepsilon_{r_{i-1}-1} \end{aligned}$$

Also

$$\begin{aligned} &\|Q_{[r_{i-1},r_i)}T\left(g_{i-1} + g_i + g_{i+1}\right) - Q_{[r_{i-1},r_i)}T\left(\bar{g}_{i-1} + \bar{g}_i + \bar{g}_{i+1}\right)\| \\ &\qquad \leq \|T\left(g_{i-1} + g_i + g_{i+1} - \bar{g}_{i-1} - \bar{g}_i - \bar{g}_{i+1}\right)\| \\ &\qquad < \varepsilon_{p_{i-1}} + \varepsilon_{p_i} + \varepsilon_{p_{i+1}} < \varepsilon_{p_{i-1}-1}. \end{aligned}$$

Finally, applying lemma 1.5 again we have

$$\begin{aligned} &\|Q_{[r_{i-1},r_i)}[\,T\left(\bar{g}_{i-1} + \bar{g}_i + \bar{g}_{i+1}\right) - T\left(\bar{x}_i\right)]\| \\ &\qquad < \varepsilon_{r_{i-1}-1}, \text{ and the subclaim follows} \end{aligned}$$

Let $\delta_i = \pm 1$. Then

$$\begin{aligned}\left\|\sum \delta_i a_i y_i\right\| &\leq \left\|\sum \delta_i \left(a_i y_i - T\bar{x}_i\right)\right\| \\ &\quad + \left\|\sum \delta_i T\bar{x}_i\right\| \\ &< \sum 4\varepsilon_{p_{i-1}-1} + \|T\| \left\|\sum \delta_i \bar{x}_i\right\| \\ &\leq 1 + C\|T\|\end{aligned}$$

The proof of Theorem A yields the following

Proposition 3.8.10. *Let X have a shrinking K-unconditional f.d.d. (E_i) and let T be a bounded linear operator from X onto Y. Let $T(CBaX) \supseteq BaY$. Then if $\varepsilon_i \downarrow 0$ and if (y_i') is a normalized weakly null basic sequence in Y there exists a subsequence (y_i) of (y_i') and integers $p_1 < p_2 < \cdots$ with the following property. Let $\|\sum a_i y_i\| \leq 2$. Then there exists $x = \sum x_i \in 2CKBaX, (x_i)$ a block basis of (E_i), such that*

$$\|Tx_i - a_i y_i\| < \varepsilon_i \quad \textit{for all } i$$

Moreover there exist (r_i) with $0 = r_0 < p_1 < r_1 < p_2 < r_2 < \cdots$ such that $x_i \in [E_j]_{j\in(r_{i-1},r_i)}$ for all i.

Corollary 3.8.11. *Let X have a shrinking K-unconditional f.d.d. and let T be a bounded linear operator from X onto the Banach space Y. Then Y contains c_0 if and only if T fixes a copy of c_0.*

Proof. If Y contains c_o then there exists (see [Ja]) (y_i), a normalized sequence in Y, with $2^{-1} \leq \|\sum a_i y_i\| \leq 2$ if $(a_i) \in S_{c_0}$, the unit sphere of c_o. Let $\varepsilon_i \downarrow 0$ with $\sum \varepsilon_i < 1$. We may assume that (y_i) satisfies the conclusion of proposition 1.9. Thus for all $n \in \mathbb{N}$ there exist

$$0 = r_0^n < p_1 < r_1^n < p_2 < r_2^n < \cdots$$

and $x_i^n \in [E_j]_{j\in\left(r_{i-1}^n, r_i^n\right)}$ such that if $x^n = \sum_{i\leq n} x_i^n$, then $\|x^n\| \leq 2CK$ and $\|Tx_i^n - y_i\| < \varepsilon_i$ for $i \leq n$.

By passing to a subsequence (x^{n_k}) we may assume $\lim_{k\to\infty} r_i^{n_k} = r_i$ and $\lim_{k\to\infty} x_i^{n_k} = x_i$ exist for all $i \in \mathbb{N}$. Thus $x_i \in [F_j]_{j\in(r_{i-1},r_i)}$ with $r_0 = 0 < r_1 < r_2 < \cdots$, $\|Tx_i - y_i\| < \varepsilon_i$ for all i and $\sup_n \|\sum_1^n x_i\| < \infty$. It follows that (x_i) is equivalent to the unit vector basis of c_0. Moreover if we choose $\omega_i \in \varepsilon_i CBaX$ with $T\omega_i = y_i - Tx_i$ then $T\left(x_i + \omega_i\right) = y_i$ and some subsequence of $(x_i + \omega_i)$ is also a c_0 basis. Hence T fixes c_0. ☐

3.8.3 The proof of Theorem B.

We begin by recalling the definition of the Schreier space $S[$ S]. Let c_{00} be the linear space of all finitely supported real valued sequences. For $x = (x_i) \in c_{00}$ set

$$\|x\| = \max_i \left\{\sum_{i=1}^p |x_{k_i}| : p \in \mathbb{N} \text{ and } p \leq k_1 < \cdots < k_p\right\}$$

S is the completion of $(c_{00}, \|\cdot\|)$. We let $\|x\|_0$ denote the c_0-norm of x. The unit vector basis (e_n) is a shrinking 1-unconditional basis for $S.S$ can be embedded into $C\left(\omega^\omega\right)$ and thus S is c_0-saturated.

Theorem B will follow from a quantitative version, Theorem B' (below). Given a sequence $(x_n), \lambda > 0$ and F a finite nonempty subset of $\mathbb{N}, y = \lambda \sum_{n \in F} x_n$ is said to be a 1-average of (x_n). We say that a Banach space X has property- $S(1)$ if every normalized weakly null sequence in X admits a block basis of 1-averages which is equivalent to the unit vector basis of $c_0.S$ has property- $S(1)$.

Theorem 3.8.12 **(B′).** *Let Y be a quotient of S. Then Y has property- $S(1)$.*

We shall use the following simple Let T be a bounded linear operator from X onto Y where X has a shrinking uncondi- Y^* is separable and so by a theorem of Zippin [Z] we may assume that Y is a subspace of a Banach space Z possessing a bimonotone shrinking basis, (z_i). Fix $C > 0$ such that

$$T\left(C B_a X\right) \supseteq B_a Y \equiv \{y \in Y : \|y\| \leq 1\}$$

Recall that $\left(\widetilde{E}_i\right)$ is a blocking of $\left(\widetilde{E}_i\right)$ if there exist integers $0 = q_0 < q_1 < q_2 < \cdots$ such that $\widetilde{E}_i = \left[\widetilde{\widetilde{E}}_j\right]_{j=q_{i-1}+1}^{q_i}$ for all i (where $[\cdots]$ denotes the closed linear span). Similarly, $\widetilde{F}_i = [z_j]_{j=q_{i-1}+1}^{q_i}$ defines a blocking of (z_i).

Fix a sequence $\varepsilon_{-1} > \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \cdots$ converging to 0 which satisfies

$$\sum_{i=-1}^{\infty} \varepsilon_i < 1/4 \text{ and } \sum_{i=p}^{\infty} (4i+2)\varepsilon_i < \varepsilon_{p-1} \text{ for } p \geq 0 \tag{1.1}$$

Then choose $\tilde{\varepsilon}_0 > \tilde{\varepsilon}_1 > \cdots$ converging to 0 which satisfies

$$4p\tilde{\varepsilon}_p < \varepsilon_{p+2} \text{ for } p \geq 1 \text{ and } \sum_{j=p+1}^{\infty} \tilde{\varepsilon}_j < \tilde{\varepsilon}_p \text{ for } p \geq 0 \tag{1.2}$$

Our first step is the blocking technique of Johnson and Zippin.

Lemma 3.8.13. *[140, 141] There exist blockings $\left(\widetilde{E}_i\right)$ and $\left(\widetilde{F}_i\right)$ of $\left(\widetilde{\widetilde{E}}_i\right)$ and (z_i), respectively, such that if $\left(\widetilde{Q}_i\right)$ is the sequence of finite rank projections on Z associated with $\left(\widetilde{F}_i\right)$ then* (1.3)

For all $i \in \mathbb{N}$ and $x \in \widetilde{E}_i$ with $\|x\| \leq C$, we have $\left\|\widetilde{Q}_j T x\right\| < \tilde{\varepsilon}_{\max(i,j)}$ if $j \neq i, i-1$.

Roughly, this says that $T\widetilde{E}_i$ is essentially contained in $\widetilde{F}_{i-1} + \widetilde{F}_i$ (where $\widetilde{F}_0 = \{0\}$). Let (y_i'') be a normalized weakly null sequence in Y. Choose a subsequence (y_i'') of (y_i) and a blocking (F_i) of $\left(\widetilde{F}_i\right)$, given by $F_i = \left[\widetilde{F}_j\right]_{j=q_{i-1}+1}^{q_i}$, such that if $Q_i = \sum_{j=q_{i-1}+1}^{q_i} \widetilde{Q}_j$ is the sequence of finite rank projections on Z associated with (F_i), then

$$\left\|Q_j y_i'\right\| < \tilde{\varepsilon}_{\max(i,j)} \text{ if } i \neq j \tag{1.4}$$

Roughly, y_i' is essentially in F_i. Furthermore we may assume that

$$\left\|\sum a_i y_i'\right\| = 1 \text{ implies } \max |a_i| \leq 2 \tag{1.5}$$

Let (E_i) be the blocking of $\left(\widetilde{E}_i\right)$ given by the same sequence (q_i) which defined (F_i), $E_i = \left[\widetilde{E}_j\right]_{j=q_{i-1}+1}^{q_i}$.

We begin with a sequence of elementary technical yet necessary lemmas.

For $I \subseteq \mathbb{N}$ we define $Q_I = \sum_{j \in I} Q_j$ and set $Q_\phi = 0$.

Lemma 3.8.14. *Let $0 \leq n < m$ be integers and let $y = \sum_{i \notin (n,m)} a_i y_i'$ with $\|y\| = 1$. Then for $j \in (n,m)$, $\|Q_j y\| < \varepsilon_j$ and $\left\|Q_{(n,m)} y\right\| < \varepsilon_n$.*

Proof. Let $n < j < m$. Then by (1.5), (1.4), (1.2) and (1.3),

$$\begin{aligned} \|Q_j y\| &\leq 2\left(\sum_{i \leq n} \left\|Q_j y_i'\right\| + \sum_{i \geq m} \left\|Q_j y_i'\right\|\right) \\ &< 2\left(n\tilde{\varepsilon}_j + \tilde{\varepsilon}_{m-1}\right) \\ &\leq (2j+2)\tilde{\varepsilon}_j \leq 4j\tilde{\varepsilon}_j < \varepsilon_j \end{aligned}$$

Thus $\left\|Q_{(n,m)} y\right\| < \sum_{j \in (n,m)} \varepsilon_j < \varepsilon_n$ by (1.1).

Lemma 1.3. Let $0 = p_0 < r_0 = 1 < p_1 < r_1 < p_2 < r_2 < \cdots$ be integers and let $y = \sum_{i=1}^{\infty} a_i y_{p_i}'$ with $\|y\| = 1$. Then for $i \in \mathbb{N}$,

$$\left\|Q_{[r_{i-1}, r_i)} y - a_i y_{p_i}'\right\| < \varepsilon_{p_{i-1}-1}$$

Proof.

$$\begin{aligned} &\left\|Q_{[r_{i-1}, r_i)} y - a_i y_{p_i}'\right\| \\ &\leq \left\|Q_{[r_{i-1}, r_i)} \sum_{j \neq i} a_j y_{p_j}'\right\| + \left\|Q_{[r_{i-1}, r_i)} a_i y_{p_i}' - a_i y_{p_i}'\right\| \end{aligned}$$

which by lemma 1.2 is

$$\begin{aligned} &< \varepsilon_{r_{i-1}-1} + \left\|Q_{[1, r_{i-1})} a_i y_{p_i}'\right\| + \left\|Q_{[r_i, \infty)} a_i y_{p_i}'\right\| \\ &< \varepsilon_{r_{i-1}-1} + 2\sum_{k < r_{i-1}} \left\|Q_k y_{p_i}'\right\| + 2\varepsilon_{r_i-1} (\text{ by (1.5) and lemma 1.2}) \\ &< \varepsilon_{r_{i-1}-1} + 2\left(r_{i-1}-1\right)\tilde{\varepsilon}_{p_i} + 2\varepsilon_{r_i-1} (\text{ by (1.4)}) \\ &\leq \varepsilon_{p_{i-1}} + 2p_i\tilde{\varepsilon}_{p_i} + 2\varepsilon_{p_i} < \varepsilon_{p_{i-1}} + 4\varepsilon_{p_i} (\text{ by (1.2)}) \\ &< \varepsilon_{p_{i-1}-1} (\text{ by 1.1}) \end{aligned}$$

□

Lemma 3.8.15. *Let $i \in \mathbb{N}, x \in E_i$ and $\|x\| \leq C$. Then*

$$\|Q_j Tx\| < \varepsilon_{\max(i,j)} \text{ if } j \neq i, i-1$$
$$\left\|Q_{[1,i-2]} Tx\right\| < \varepsilon_{i-1} \text{ and } \left\|Q_{[i,\infty)} Tx\right\| < \varepsilon_{i-1}$$

Proof. Let $x = \sum_{\ell \in (q_{i-1}, q_i]} \omega_\ell$ with $\omega_\ell \in \widetilde{E}_\ell$.

$$\begin{aligned}
\|Q_j Tx\| &\leq \sum_{k \in (q_{j-1}, q_j]} \sum_{\ell \in (q_{i-1}, q_i]} \left\|\widetilde{Q}_k T \omega_\ell\right\| \\
(\text{ if } j < i-1) : &< \sum_{k \in (q_{j-1}, q_j]} \sum_{\ell \in (q_{i-1}, q_i]} \tilde{\varepsilon}_\ell (\text{ by } (1.4)) \\
&< q_j \tilde{\varepsilon}_{q_{i-1}} < q_{i-1} \tilde{\varepsilon}_{q_{i-1}} < \varepsilon_{q_{i-1}+2} < \varepsilon_i \text{ using } (1.2) \\
(\text{ if } j > i) : &< \sum_{k \in (q_{j-1}, q_j]} \sum_{\ell \in (q_{i-1}, q_i]} \tilde{\varepsilon}_k \\
&< \sum_{k \in (q_{j-1}, q_j]} q_i \tilde{\varepsilon}_k \leq q_i \tilde{\varepsilon}_{q_{j-1}} \\
&\leq q_{j-1} \tilde{\varepsilon}_{q_{j-1}} < \varepsilon_{q_{j-1}+2} \leq \varepsilon_{j+1} < \varepsilon_j .
\end{aligned}$$

Finally,

$$\left\|Q_{[1,i-2]} Tx\right\| \leq \sum_{k=1}^{i-2} \|Q_k Tx\| < \sum_{k=1}^{i-2} \varepsilon_i = (i-2)\varepsilon_i < \varepsilon_{i-1}$$

and

$$\left\|Q_{[i,\infty)} Tx\right\| \leq \sum_{k=i}^{\infty} \|Q_k Tx\| < \sum_{k=i}^{\infty} \varepsilon_k < \varepsilon_{i-1}$$

□

Lemma 3.8.16. *Let (x_n) be a normalized weakly null sequence in S with $\lim_n \|x_n\|_0 = 0$. Then some subsequence of (x_n) is equivalent to the unit vector basis of c_0.*

Let T be a bounded linear operator from S onto a Banach space Y and let (y_i') be a normalized weakly null basic sequence in Y. Let $T(CBaS) \supseteq BaY$.

Lemma 3.8.17. *If no block basis of 1-averages of (y_i') is equivalent to the unit vector basis of c_0, then there exists $\delta > 0$ such that if $x \in 3CBaS, Tx$ is a 1-average of (y_i') and $\|Tx\| > 1/3$ then $\|x\|_0 > \delta$.*

Proof. If no such δ exists then there exists $(x_i) \subseteq 3CBaS$ with $\lim_i \|x_i\|_0 = 0, \|Tx_i\| > \frac{1}{3}$ and Tx_i a 1-average of (y_i') for all i. By lemma 2.1 there exists a subsequence (x_i') of (x_i) which is equivalent to the unit vector basis of c_0. By passing to a further subsequence we may assume that (Tx_i') is a seminormalized weakly null basic sequence in $[(y_i')]$. Thus (Tx_i') is also equivalent to the unit vector basis of c_0. □

Proof of Theorem B'. Let (y'_i) be a normalized weakly null sequence in Y. If (y'_i) fails the $S(1)$ property, choose $\delta > 0$ by lemma 2.1. Let $(\varepsilon_i)_{i=1}^{\infty}$ be a sequence of positive numbers satisfying

$$\sum_{i=1}^{\infty} \varepsilon_i < \min(\delta/(2C), 1) \tag{2.1}$$

Let (y_i) be the subsequence of (y'_i) given by proposition 1.9 for the sequence (ε_i).

Choose an even integer $m \in \mathbb{N}$ with

$$m > 8C/\delta \tag{2.2}$$

From the theory of spreading models there exists $(z_i)_{i=1}^{2m}$, a finite subsequence of (y_i), such that setting $\lambda = \left\| \sum_{i=1}^{2m} z_i \right\|^{-1}$,

$$2 > \lambda \left\| \sum_{i \in F} z_i \right\| > 1/3 \tag{2.3}$$

whenever $F \subseteq \{1, \cdots, 2m\}$ with $|F| \geq m$.

Thus there exists $x = \sum_{i=1}^{2m} x_i \in 2CBaS$ with (x_i) a block basis of (e_i) and $\|Tx_i - \lambda z_i\| < \varepsilon_i$ for $i \leq 2m$. For $i \leq 2m$ choose $\omega_i \in S$ with $T\omega_i = \lambda z_i - Tx_i$ and $\|\omega_i\| \leq C\varepsilon_i$. Hence $T(x_i + \omega_i) = \lambda z_i$.

Since $\left\| T\left(\sum_1^{2m} (x_i + \omega_i) \right) \right\| > 1/3$, and

$$\left\| \sum_1^{2m} (x_i + \omega_i) \right\| \leq \left\| \sum_1^{2m} x_i \right\| + \sum_1^{2m} \|\omega_i\| < 2C + \sum_1^{\infty} \varepsilon_i C < 3C$$

by lemma 2.2 we have $\left\| \sum_1^{2m} (x_i + \omega_i) \right\|_0 > \delta$. Since $\left\| \sum_1^{2m} \omega_i \right\|_0 \leq \left\| \sum_1^{2m} \omega_i \right\| < \delta/2$ by (2.1) there exists $i_1 \leq 2m$ with $\|x_{i_1}\|_0 > \delta/2$.

Now

$$\left\| T\left(\sum_{\substack{i=1 \\ i \neq i_1}}^{2m} (x_i + \omega_i) \right) \right\| = \left\| \sum_{\substack{i=1 \\ i \neq i_1}}^{2m} \lambda z_i \right\| > \frac{1}{3}$$

and so we may repeat the argument above finding $i_2 \neq i_1$ with $\|x_{i_2}\|_0 > \delta/2$. In fact by (2.3) we can repeat this m-times obtaining distinct integers $(i_k)_{k=1}^{m} \subseteq \{1, 2, \cdots, 2m\}$ with $\|x_{i_k}\|_0 > \delta/2$ for $k \leq m$. But then $2C \geq \|x\| = \left\| \sum_{i=1}^{2m} x_i \right\| \geq \|\sum_{k=1}^{m} x_{i_k}\| \geq \sum_{k=\frac{m}{2}+1}^{m} \|x_{i_k}\|_0 \geq \delta m/4$ which contradicts (2.2).

□

3.8.4 Open Problems

Our work suggests a number of problems, of which we list a few. For a more extensive list of related problems and an overview of the current state of infinite dimensional Banach space theory, see [*R*].

Problem 1. Let X be a Banach space having property (WU) which does not contain ℓ_1 and let Y be a quotient of X. Does Y have property (WU)?

In light of Theorem A it is worth noting that $C\left(\omega^\omega\right)$ has property (WU) [MR] but does not embed into any space having a shrinking unconditional f.d.d. In fact $C\left(\omega^\omega\right)$ is not even a subspace of a quotient of such a space. Indeed $C\left(\omega^\omega\right)$ fails property (U) (see e.g. [HOR]) while any quotient of a space with a shrinking unconditional f.d.d. will have property (U). In fact if X has property (U) and does not contain ℓ_1, then any quotient of X will have property (U) [142]. The next problem is due to H. Rosenthal.

Problem 2. Let X have a shrinking unconditional f.d.d. and let Y be a quotient of X. Does Y embed into a Banach space having a shrinking unconditional f.d.d.?

We say that a Banach space Y has uniform-(WU) if there exists $K < \infty$ such that every normalized weakly null sequence in Y has a K-unconditional subsequence. Our proof of Theorem A showed that the quotient space Y has uniform-(WU).

Problem 3. If Y has property (WU) does Y have uniform-(WU)?

Theorem B solved a special case of the following well known problem.

Problem 4. Let Y be a quotient of $C\left(\omega^\omega\right)$ (or more generally $C(K)$ where K is a compact countable metric space). Is $Y c_0$-saturated?

Regarding this problem, T. Schlumprecht [143] has observed that if Y is a quotient of $C\left(\omega^\omega\right)$, then the closed linear span of any normalized weakly null sequence in Y which has ℓ_1 as a spreading model must contain c_0.

It is not true that the quotient of a c_0-saturated space must also be c_0-saturated. The separable Orlicz function space $H_M(0,1)$, with $M(x) = \left(e^{x^4} - 1\right)/(e-1)$, considered in [CKT] is c_0-saturated and yet has ℓ_2 as a quotient. We wish to thank S. Montgomery-Smith for bringing this fact to our attention. However this space does not have an unconditional basis and so we ask

Problem 5. Let X be a c_0-saturated space with an unconditional basis and let Y be a quotient of X. Is $Y c_0$-saturated?

A more restricted and perhaps more accessible question is the following **problem 6.** let$^n th$-schreier space, where $n \geq 2$. Is Y, c_0 saturated? Does Y have a property - s_n ?
s_n is defined as follows. let $\|x\|_{n+1} = \max\{\sum_{k=1}^{p} \|E_k x\|_n : p < E_1 < E_2 < \cdots < E_n\}$.
Here $p \leq E_1$ means $p \leq min E_1$ and $E_1 < E_2$ means $\max E_1 < \min E_2$. Also $Ex(i) = x(i); i \in E,$ 0otherwise. S_{n+1} is the completion of $(c_{00}, \|.\|_{n+1})$. The unit vector basis (e_n) is a 1-unconditional shrinking basis for every S_n and S_n embeds into $C(\omega^{\omega^n})$.

Property-S(n) is defined as follows. n-averages of a sequence (y_m) are defined inductively: an $n+1$-average of (y_m) is a 1-average of a block basis of normalized n-averages. Y has property-$S(n)$ if every normalized weakly null basic sequence in Y admits a block basis of n-averages equivalent to the unit vector basis of c_0. S_n has property-$S(n)$.

3.9 The proportional UAP characterizes weak Hilbert spaces

[†]

W. B. Johnson and G. Pisier

Abstract: We prove that a Banach space has the uniform approximation property with proportional growth of the uniformity function iff it is a weak Hilbert space.

3.9.1 Introduction.

The "weak Hilbert spaces" were introduced and studied in [144]. Among the many equivalent characterizations in [144] perhaps the simplest definition is the following. A Banach space is a weak Hilbert space if there is a constant C such that for all n, for all n-tuples $(x_1, \cdots, x_n)$ and $(x_1^*, \cdots, x_n^*)$ in the unit balls of X and X^* respectively, we have

$$|\det(< x_i^*, x_j >)|^{\frac{1}{n}} \leq C.$$

The first example of a non Hilbertian weak Hilbert space was obtained by the first author (cf. [146], Example 5.3 and [145]). Recall that a Banach space X has the uniform approximation property (in short UAP) if there is a constant K and a function $n \to f(n)$ such that for all n and all n-dimensional subspaces $E \subset X$, there is an operator $T : X \to X$ with $rk(T) \leq f(n)$ such that $\|T\| \leq K$ and $T_{|E} = I_{|E}$. For later use, given $K > 1$ we introduce

$$k_X(K, n) = \sup_{\substack{E \subset X \\ \dim E = n}} \quad \inf \{rk(T)\}$$

where the infimum runs over all $T : X \to X$ such that $\|T\| \leq K$ and $T_{|E} = I_{|E}$. Note that X has the UAP iff there is a constant K such that $k_X(K, n)$ is finite for all n; we then say that X has the K-UAP. The asymptotic growth of the function $n \to k_X(K, n)$ provides a quantitative measure of the UAP of the space X.

For instance, if X is a Hilbert space we have clearly $k(1, n) = n$, hence if X is isomorphic to a Hilbert space there is a constant K such that

$$k_X(K, n) = n \quad \text{for all } n.$$

The converse is also true by the complemented subspace theorem of Lindenstrauss-Tzafriri [147].

The main result in this paper can be viewed as an analogous statement for weak Hilbert spaces, as follows.

Theorem 3.9.1 (Main Theorem.). *A Banach space X is a weak Hilbert space iff there are constants K and C such that*

$$k_X(K, n) \leq Cn \quad \text{for all } n. \tag{0.1}$$

[†] Both authors were supported in part by NSF DMS 87-03815

That is, proportional asymptotic behavior of the uniformity function in the definition of the UAP characterizes weak Hilbert spaces. It was proved in [144] that weak Hilbert spaces have the UAP but no estimate of the function $n \to k_X(K, n)$ was obtained. For the purposes of this paper we will say that X has the proportional UAP if there are constants K and C such that (0.1) holds. The authors thank V. Mascioni and G. chechtman for several discussions concerning the material in this paper.

3.9.2 Weak Hilbert spaces have proportional UAP

We first recall a characterization of weak Hilbert spaces in terms of nuclear operators. Recall that an operator $u : X \to X$ is called nuclear if it can be written

$$u(x) = \sum_{n=1}^{\infty} x_n^*(x) x_n$$

with $x_n^* \in X^*, x_n \in X$ such that $\sum \|x_n^*\| \; \|x_n\| < \infty$. Moreover the nuclear norm $N(u)$ is defined as

$$N(u) = \inf \left\{ \sum \|x_n^*\| \; \|x_n\| \right\}$$

where the infimum runs over all possible representations. We also recall the notation for the approximation numbers

$$\forall k \geq 1 \quad a_k(u) = \inf \{\|u - v\| \;\; | \; v : X \to X, \;\; rk(v) < k\}.$$

By [144], a Banach space X is a weak Hilbert space iff there is a constant C such that for all nuclear operators $u : X \to X$ we have

$$\sup_{k \geq 1} \; k a_k(u) \leq C N(u) \tag{1.1}$$

The following observation is identical to reasoning already used by V. Mascioni [148].

Proposition 3.9.2. *Let X be a weak Hilbert space. Assume that there is a constant K' such that for all n and all n dimensional subspaces $E \subset X$ there is an operator $u : X \to X$ such that $u_{|E} = I_{|E}$, $\|u\| \leq K'$ and $N(u) \leq K'n$. Then X has the proportional UAP. (Recall that if u has finite rank then $N(u) \leq rk(u)\|u\|$, hence the converse to the preceding implication is obvious.)*

Proof. Let u be as in the preceding statement. We use (1.1) with $k = [2CK'n] + 1$, so that

$$a_k(u) \leq C N(u) k^{-1} \leq C K' n k^{-1} \leq \frac{1}{2}.$$

This means that there is an operator $v : X \to X$ with $rk(v) \leq 2CK'n$ such that $\|u - v\| \leq \frac{1}{2}$. By perturbation, it follows that the operator

$$V = v - u + I$$

is invertible on X with $\|V^{-1}\| \leq 2$. Moreover we have

$$V_{|E} = v_{|E}. \tag{1.2}$$

It follows that if we let $T = V^{-1}v$, then we have

$$\|T\| \leq \|V^{-1}\| \ \|v\| \leq 2(\|u\| + \|u - v\|) \leq 2K' + 1,$$

also $rk(T) \leq rk(v) \leq 2CK'n$ and $T_{|E} = I_{|E}$ by (1.2).

We conclude that X has the UAP with $k_X(K,n) \leq 2CK'n$, where $K = 2K' + 1$. $\square$

We will use duality via the following proposition (a similar kind of criterion was used by Szankowski [149] to prove that certain spaces fail the UAP):

Proposition 3.9.3. *Let X be a reflexive Banach space with the approximation property; in short, AP; let α, β be positive constants; and let $n \geq 1$ be an integer. The following are equivalent.*

(i) For all nuclear operators T_1, T_2 on X such that $T_1 + T_2$ has rank $\leq n$, we have

$$|tr(T_1 + T_2)| \leq \alpha N(T_1) + \beta n \|T_2\|.$$

(ii) Same as (i) with T_1, T_2 of finite rank.

(iii) For any subspace $E \subset X$ with dimension $\leq n$ there is an operator $u : X \to X$ such that $u_{|E} = I_E$, $\|u\| \leq \alpha$ and $N(u) \leq \beta n$.

Proof. (i) $\Rightarrow$ (ii) is trivial.

Assume that (ii) holds.

We equip $X^* \otimes X$ with the norm $|w| = \inf \{\alpha N(T_1) + \beta n \|T_2\|\}$ where the infimum runs over all decompositions

$$u = T_1 + T_2 \text{ with } T_1 \text{ and } T_2 \text{ in } X^* \otimes X$$

(identified with the set of finite rank operators on X). On $X^* \otimes X$ this norm is clearly equivalent to the operator norm.

Now let $E \subset X$ be a fixed subspace with $\dim(E) \leq n$. Let $\mathcal{S} \subset X^* \otimes X$ be the subspace $X^* \otimes E$ of all the operators on X with range in E. On this linear subspace the linear form ξ defined by $\xi(w) = tr(w)$ has norm ≤ 1 relative to $|\cdot|$ by our assumption (ii).

Therefore there is a Hahn-Banach extension $\tilde{\xi}$ defined on the whole of $X^* \otimes X$ which extends ξ and satisfies

$$|\tilde{\xi}(w)| \leq |w| \quad \forall w \in X^* \otimes X. \tag{1.3}$$

By classical results, $\tilde{\xi}$ can be identified with an integral operator $u : X \to X^{**}$. Since X is reflexive, u is actually a nuclear operator on X, and we have $\tilde{\xi}(w) = tr(wu)$ for all w in $X^* \otimes X$.

Since $\tilde{\xi}$ extends ξ, we must have

$$\forall x^* \in X^* \quad \forall e \in E \quad < \tilde{\xi},\ x^* \otimes e >= tr(x^* \otimes e) = x^*(e) \quad \text{hence } x^*(ue) = x^*(e).$$

Equivalently

$$u_{|E} = I_{|E}.$$

On the other hand, by (1.3) we have

$$|tr(uT_1)| \leq \alpha N(T_1) \quad \text{and } |tr(uT_2)\ | \ \leq \beta n\|T_2\|$$

for all finite rank operators T_1 and T_2 on X.

This implies $\|u\| \leq \alpha$ and (again using the reflexivity of X) $N(u) \leq \beta n$.

This shows that (ii)⇒(iii).

Finally we show (iii)⇒(i). Assume (iii). Let T_1, T_2 be as in (i), let E be the range of $T_1 + T_2$ and let u be as in (iii). Then we have $T_1 + T_2 = u(T_1 + T_2)$ hence since X has the AP (which ensures that $|tr(T)| \leq N(T)$ for all nuclear operator $T : X \to X$) we have $|tr(T_1 + T_2)| = |tr(uT_1) + tr(uT_2)|$

$$\begin{aligned} &\leq \|u\|N(T_1) + N(u)\|T_2\| \\ &\leq \alpha(T_1) + \beta n\|T_2\|. \end{aligned}$$

□

Remark 35. *Note that (i) is also equivalent to (i'): (i') For all T_1, T_2 on X such that (T_1+T_2) has rank $\leq n$, we have*

$$N(T_1 + T_2) \leq \alpha N(T_1) + \beta n\|T_2\|.$$

Indeed; (assuming the AP and reflexivity) we have

$$N(T_1 + T_2) = \sup\{tr(S(T_1 + T_2))\ ;\ S : X \to X,\ \|S\| \leq 1\}.$$

This shows that (i)⇒ (i'). Since X has the AP the converse is obvious. Of course, a similar remark holds for (ii).

Remark 36. *If X is not assumed to have the AP a variant of Proposition 1.2 will still hold provided we use the projective tensor norm on $X^* \otimes X$ instead of the nuclear norm.*

We will use the following result already exploited in [144] to prove that weak Hilbert spaces have the AP. Whenever $u : X \to X$ is a finite rank operator, we denote by $\det(I + u)$ the quantity

$$\Pi(1 + \lambda_j(u))$$

where $\{\lambda_j(u)\}$ are the eigenvalues of u repeated according to their algebraic multiplicity. Equivalently, $\det(I+u)$ is equal to the ordinary determinant of the operator $(I+u)_{|E}$ restricted to any finite dimensional subspace $E \subset X$ containing the range of u.

Lemma 3.9.4. *Let u, v be finite rank operators on a weak Hilbert space X with $rk(u) \leq n$. Then we have*

$$|\det\ (I + u + v)| \leq \Big(\sum_{j=0}^{n} \frac{C^j}{j!} N(u)^j\Big) \exp CN(v) \tag{1.4}$$

where C is the "weak Hilbert space constant" of X; that is to say,

$$C = \sup_{\substack{x_i \in B_X \\ x_i^* \in B_X^*}} |\det(< x_i^*, x_j >)|^{1/n}. \tag{1.5}$$

For the proof we refer to [150, p.229]. Note that if $N(u) \geq 1$ then (1.4) implies for all complex numbers z,

$$\det\,(I + z(u+v)) \leq N(u)^n \exp\{C|z| + C|z|N(v)\}. \tag{1.6}$$

Let $f(z) = \det(I + z(u+v))$. Then f is a polynomial function of $z \in \mathbb{C}$ such that

$$f(0) = 1 \text{ and } f'(0) = tr(u+v).$$

In [151], Grothendieck showed that the function $u \to \det\,(I+u)$ is uniformly continuous on $X^* \otimes X$ equipped with the projective norm, and therefore extends to the completion $X^* \hat{\otimes} X$. This shows that if X has the AP, the determinant $\det\,(I+v)$ can be defined unambiguously for any nuclear operator v on X. As shown in [151], the function $z \to \det\,(I + z(u+v))$ is an entire function satisfying (1.4) if u is of rank $\leq n$ and v possibly of infinite rank. We use this extension in Theorem 1.5 below, but in the proof of our main result the special case of v of finite rank in Theorem 1.5 is sufficient. This makes our proof more elementary.

We will make crucial use of the following classical inequality of Carathéodory; we include the proof for the convenience of the reader.

Lemma 3.9.5. *Let h be an analytic function in a disc $D_R = \{z \in \mathbb{C}\ ;\quad |z| < R\}$ such that $h(0) = 0$. Then for any $0 < r < R$ we have*

$$|h'(0)| \leq \frac{2}{r} \sup_{|z|=r} \Re(h(z)).$$

Proof. Let $M = \sup\{\Re(h(z)), |z| < r\}$. Note that $M \geq 0$.

Let $g(z) = \frac{h(z)}{2M - h(z)}$. Then $|g(z)| \leq 1$ if $|z| \leq r$, g is analytic in D_r and $g(0) = 0$. By the Schwarz lemma we have

$$|g(z)| \leq \frac{|z|}{r} \text{ for all } z \text{ in } D_r \text{ and } |g'(0)| \leq 1/r.$$

Since $h(z) = \frac{2Mg(z)}{1+g(z)}$ we have $h'(0) = 2Mg'(0)$ hence $|h'(0)| \leq 2M/r$.

□

We now prove the main result of this section, namely that any weak Hilbert space has the proportional UAP. Let X be a weak Hilbert space. We will show that X satisfies (ii) in Proposition 1.2. Actually, we obtain the following result of independent interest.

Proof. Let X be a weak Hilbert space with constant C as in (1.5). Let u, v be nuclear operators on X and let ρ be the spectral radius of $u+v$. Then if $rk(u) \leq n$ and $N(u) > 1$

$$|tr(u+v)| \leq 2n\rho\ Log N(u) + 2C + 2CN(v) \tag{1.7}$$

hence also

$$N(u+v) \leq 2n\|u+v\| LogN(u) + 2C + 2CN(v). \tag{1.8}$$

□

Proof. Let $R = 1/\rho$. The function $f(z) = \det(I + z(u+v))$ is entire and does not vanish in D_R. Therefore there is an analytic function h on D_R such that $f = \exp(h)$ and since $f(0) = 1$ we can assume $h(0) = 0$.

Note that $f'(0) = h'(0) = tr(u+v)$. By (1.6) we have if $N(u) \geq 1$ and $r < R$

$$\sup_{|z|=r} \Re\, h(z) \leq nLogN(u) + Cr + CrN(v)$$

hence by Lemma 1.4

$$|tr(u+v)| = |h'(0)| \leq \frac{2n}{r} LogN(u) + 2C + 2CN(v)$$

Letting r tend to $R = \frac{1}{\rho}$, we obtain (1.7).

For (1.8) we simply observe that if $N(u) > 1$ we have

$$N(u) = \sup\{|tr(uS)|; \quad S : X \to X, \quad \|S\| \leq 1, \quad N(uS) > 1\}. \tag{1.9}$$

Therefore (1.8) follows from (1.7) since $\rho \leq \|u+v\|$ and we can take the supremum of (1.7) over all S as in (1.9). □

Finally we prove the "only if" part of our main theorem. Let X be a weak Hilbert space. The first and second authors proved, respectively, that X is reflexive (cf. [150], chapter 14) and that X has the AP ([150], chapter 15). We will show that (ii) in Proposition 1.2 holds for suitable constants. Let T_1, T_2 be as in Proposition.

Proposition 3.9.6. *Let $u = T_1 + T_2$ and $v = -T_1$. By homogeneity we may assume $n\|T_2\| + N(T_1) = 1$.*

Then if $N(T_1 + T_2) > 1$ we have by (1.8)

$$\begin{aligned} N(T_1+T_2) &\leq N(u+v) + N(v) \\ &\leq 2n\|T_2\| LogN(T_1+T_2) + 2C + (2C+1)N(T_1) \\ &\leq 2LogN(T_1+T_2) + 4C + 1, \end{aligned}$$

and (since $2Logx \leq (x/2) + 2$ if $x > 1$) this implies that if $N(T_1+T_2) > 1$ then

$$N(T_1+T_2) \leq 8C + 6.$$

Since in the case $N(T_1+T_2) \leq 1$ this bound is trivial, we conclude by homogeneity that (if $T_1 + T_2$ has rank $\leq n$)

$$N(T_1+T_2) \leq (8C+6)(n\|T_2\| + N(T_1)).$$

By proposition 1.2 and 1.1 we conclude that X has the proportional UAP.

Remark 37. *Replacing* $(u+v)$ *by* $\epsilon(u+v)$ *in (1.7) yields that if* $\epsilon \geq N(u)^{-1}$*, then* $|tr(u+v)| \leq 2n\rho Log(\epsilon N(u)) + 2C\epsilon^{-1} + 2CN(v)$ *hence after minimization over* $\epsilon \geq N(u)^{-1}$ *we find that if* $N(u) \geq n\rho/C$*, then*

$$|tr(u+v)| \leq 2n\rho(Log^{+}(\frac{CN(u)}{n\rho})+1) + 2CN(v).$$

On the other hand if $N(u) < n\rho/C$ we have trivially since $C \geq 1$

$$\begin{aligned} |tr(u+v)| &\leq N(u+v) \\ &\leq n\rho/C + N(v) \\ &\leq 2n\rho + 2CN(v) \end{aligned}$$

hence we conclude that without any restriction on $N(u)$ we have if $rk(u) \leq n$

$$|tr(u+v)| \leq 2n\rho(^{+}(\frac{CN(u)}{n\rho})+1) + 2CN(v).$$

Even in the case of a Hilbert space we do not see a direct proof of this inequality.

3.9.3 The converse.

Recall that X is a weak cotype 2 space if there are constants C and $0 < \delta < 1$ such that every finite dimensional subspace $E \subset X$ contains a subspace $F \subset E$ with $\dim F \geq \delta \dim E$ such that $d_F \equiv d(F, l_2^{\dim F}) \leq C$ (cf. [152]). We begin with a slightly modified presentation of Mascioni's [153] proof that a Banach space X which has proportional UAP must have weak cotype 2. Suppose that $k_X(n,K) \leq Ln$ for all $n = 1, 2, \ldots$. Take any $(1+\delta)n$-dimensional subspace G_0 of X, and, using Milman's subspace of quotient theorem[154] (or see [150], chapters 7 & 8), choose an n–dimensional subspace G of G_0 for which there exists a subspace H of G such that $\dim H \leq \delta n$ and $d = d_{G/H}$ is bounded by a constant which is independent of n, where δ is chosen so that $\delta L \leq \frac{1}{2}$.

Take $T\colon X \to X$ with $T_{|H} = I_H$, $\|T\| \leq K$, and $rk(T) \leq \delta L n$. Let $Q\colon G \to G/H$ be the quotient map and set $E = \ker(T) \cap G$. If x is in E, then

$$\begin{aligned} \|Qx\| &= \inf_{h\in H} \|x-h\| \\ &\geq \inf_{h\in H} \frac{\|(I-T)(x-h)\|}{\|I-T\|} \\ &= \frac{\|x\|}{\|I-T\|}\,; \end{aligned}$$

that is, $Q_{|E}$ is an isomorphism and $\|(Q_{|E})^{-1}\| \leq \|I-T\|$. Thus $d_E \leq \|I-T\| d_{G/H}$, which finishes the proof since $\dim E \geq n - \delta L n \geq \frac{n}{2}$.

Since we do not know *a priori* that the proportional UAP dualizes, we need to prove Mascioni's theorem under a weaker hypothesis.

Theorem 3.9.7. *Let* X *be a Banach space. Assume that there are constants* K *and* L *such that for all finite dimensional subspaces* $E \subset X$ *there is an operator* $T : X \to X$ *satisfying* $T_{|E} = I_E$ *and such that* $\|T\| \leq K$ *and* $\pi_2(T) \leq L(\dim E)^{1/2}$*. Then* X *is a weak cotype 2 space.*

Proof:

Proof. Recall that if $T : X \to Y$ is an operator, then

$$\pi_2(T)^2 = \sup\left\{\sum \|Tu(e_i)\|^2; u : l_2 \to X,\ \|u\| \le 1\right\}$$

where $\{e_i\}_{i=1}^{\infty}$ is the unit vector basis for l_2.

Take any $(1+\delta)n$-dimensional subspace G_0 of X, where δ is chosen so that $L\sqrt{\delta} \le \frac{1}{8}$. By Milman's subspace of quotient theorem [154] or [150], chapters 7 & 8), we can choose an n–dimensional subspace G of G_0 for which there exists a subspace H of G such that $\dim(H) \le \delta n$ and $d = d_{G/H}$ is bounded by a constant which is independent of n. Using the ellipsoid of maximal volume, we get from the Dvoretzky-Rogers lemma (cf. [150], lemma 4.13) a norm one operator $J\colon l_2^n \to G$ such that $\|x_i\| \ge \frac{1}{2}$ for all $i = 1, 2, \ldots, \frac{n}{2}$, where $x_i = Je_i$. By [146], all we need to check is that there is a constant τ so that

$$\left(\text{Average}_{\pm} \|\sum_{i=1}^{n} \pm x_i\|^2\right)^{\frac{1}{2}} \ \ge\ \tau\sqrt{n}.$$

Let $Q\colon G \to G/H$ be the quotient map. Then

$$\tau_0\sqrt{n/4}\ \equiv\ \left(\text{Average}_{\pm} \|\sum_{i=1}^{n/2} \pm Qx_i\|^2\right)^{\frac{1}{2}} \ \ge\ \frac{1}{d}\left(\sum_{i=1}^{n/2} \|Qx_i\|^2\right)^{\frac{1}{2}}.$$

So we can assume without loss of generality that $\|Qx_i\| \le d\tau_0$ for $1 \le i \le \frac{n}{4}$. Now take $T\colon X \to X$ with $T_{|H} = I_H$ and $\pi_2(T) \le L\sqrt{\delta n}$. Thus also

$$\left(\sum_{i=1}^{n/4} \|Tx_i\|^2\right)^{\frac{1}{2}} \ \le\ L\sqrt{\delta n}$$

hence without loss of generality $\|Tx_1\| \ \le\ 2L\sqrt{\delta}$. But then

$$\begin{aligned} d\tau_0 \ \ge\ \ & \|Qx_1\| \quad = \inf_{h\in H} \|x_1 - h\| \\ \ge\ \ & \frac{\|(I-T)x_1\|}{\|I-T\|} \quad \ge\ \frac{\frac{1}{2} - \|Tx_1\|}{\|I-T\|} \\ \ge\ \ & \frac{\frac{1}{2} - 2L\sqrt{\delta}}{\|I-T\|} \quad \ge\ \frac{1}{4\|I-T\|}; \end{aligned}$$

that is, $\tau_0 \ \ge\ (4d\|I-T\|)^{-1}$. □

Proof of converse of Main Theorem: By[155], X^{**} also has proportional UAP; in fact, $k_{X^{**}}(K,n) = k_X(K,n)$ for all K and n. Then, just as in the proof of Theorem 4 in [148], Lemma 1 in[148] (or its unpublished predecessor proved by Bourgain and mentioned in [148]) yields that X^* satisfies the hypothesis of Theorem 2.1 and hence X^* as well as X has weak cotype 2. By the results of [156], it only remains to check that X has non-trivial type; this is done as in Theorem 3.3 of

[153]: since X and X^* have weak cotype 2, they both have cotype $2+\epsilon$ for all $\epsilon > 0$. Since X has the bounded approximation property, the main result of [156] yields that X has non-trivial type.

□

Remark 38. *With a bit more work, the converse of the Main Theorem can be improved. Following Mascioni [148], given an ideal norm* α*, a normed space* X*, and* $K > 1$*, we write*

$$\alpha{-}k_X(K,n) = \sup_{\substack{E \subset X \\ \dim E = n}} \inf \{\alpha(T)\}$$

where the infimum runs over all finite rank operators $T : X \to X$ *such that* $\|T\| < K$ *and* $T_{|E} = I_{|E}$.

(We use "$< K$" instead of "$\leq K$" in order to avoid in the sequel statements involving awkward "$K+\epsilon$ for all $\epsilon > 0$".) We say that X has the α-UAP if there is a $K > 1$ such that for all n, α-$k_X(K,n) < \infty$; when the value of K is important, we say that X has the K-α-UAP. Notice that the "finite rank" can be ignored if the space X has the metric approximation property or (by adjusting K) if X has the bounded approximation property. Here we are interested in $\alpha = \pi_2$ and $\alpha = \pi_2^d$, where $\pi_2^d(T) \equiv \pi_2(T^*)$. Since for either of these α's, $\alpha(T) < \infty$ implies that T^2 is uniformly approximable by finite rank operators (T^2 factors through a Hilbert-Schmidt operator), for these two α's the K-α-UAP implies the bounded approximation property. In fact, by passing to ultraproducts and using[157], it follows that for either of these α's the K-α-UAP implies the $(K^2+\epsilon)$-UAP; in particular, X^{**} has the bounded approximation property. (This is really a sloppy version of Mascioni's reasoning [148]; Mascioni gives a better estimate for $k_X(K',n)$ in terms of α-$k_X(K,n)$.) We now state an improvement of the converse in the Main Theorem:

Theorem 3.9.8. *Suppose that there are constants K and L so that the Banach space X satisfies for all n* π_2-$k_X(K,n) \leq L\sqrt{n}$ *and* π_2^d-$k_X(K,n) \leq L\sqrt{n}$*. Then X is a weak Hilbert space.*

Proof. In view of the discussion above, we can ignore the "finite rank" condition in the definition of α-UAP. It is then easy to see for $\alpha = \pi_2$ or $\alpha = \pi_2^d$ that for all n and K α-$k_{X^{**}}(K,n) = \alpha$-$k_X(K,n)$, hence by Lemma 1 of [148] and Theorem 2.1 we conclude that X and X^* have weak cotype 2. The argument used in the proof of the converse in the Main Theorem shows that X has non-trivial type, so X is weak Hilbert. □

3.9.4 Related results and concluding remarks

In [153], Mascioni proved (but stated in slightly weaker form) that for $2 < p < \infty$ and all K, there exists $\delta = \delta(p,K) > 0$ so that for all n, $k_p(K,n) \geq \delta n^{p/2}$. (We write $k_p(K,n)$ for $k_{L_p}(K,n)$ and α-$k_p(K,n)$ for α-$k_{L_p}(K,n)$. See [153, 158, 159] for results about the UAP in L_p-spaces, $1 \leq p \leq \infty$.) We prove here the corresponding result for $1 < p < 2$.

Theorem 3.9.9. *For each* $2 < p < \infty$ *and* $K > 1$*, there exists* $\epsilon = \epsilon(p,K) > 0$ *so that for all n,* π_2-$k_p(K,n) \geq \epsilon n^{\frac{p}{4}}$*. Consequently, for* $1 < q < 2$*,* $k_q(K,n) \geq \pi_2^d$-$k_q(K,n)^2 \geq \epsilon^2 n^{\frac{p}{2}}$ *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. The proof is basically the same as the proof of Theorem 2.1 once we substitute a result of Gluskin[160] for Milman's subspace of quotient theorem, so we use notation similar to that used in Theorem 2.1. Fix n, set $G = l_p^n$, let J denote the formal identity from l_2^n into l_p^n, and let $x_i = Je_i$ be the unit vector basis of l_p^n. By Gluskin's theorem [160], given any $\gamma > 0$ there is $M = M(p, \gamma)$ independent of n and a subspace H of G with $\dim H \leq Mn^{\frac{2}{p}}$ so that $d \equiv d_{G/H} \leq \gamma n^{\frac{1}{2}-\frac{1}{p}}$. Let $Q\colon G \to G/H$ be the quotient map. Define ϵ_0 by the formula π_2-$k_p(K, \dim H) = \epsilon_0(\dim H)^{\frac{p}{4}}$ and choose $T\colon X \to X$ with $T_{|H} = I_H$, $\|T\| \leq K$, and $\pi_2(T) \leq \epsilon_0 M^{\frac{p}{4}}\sqrt{n}$.

We need to show that ϵ_0 is bounded away from 0 independently of n. Now

$$n^{\frac{1}{p}} = \left(\text{Average}_{\pm} \|\sum_{i=1}^{n} \pm x_i\|^2\right)^{\frac{1}{2}} \geq \left(\text{Average}_{\pm} \|\sum_{i=1}^{n} \pm Qx_i\|^2\right)^{\frac{1}{2}} \geq \frac{1}{d}\left(\sum_{i=1}^{n} \|Qx_i\|^2\right)^{\frac{1}{2}},$$

So we can assume without loss of generality that for $i = 1, \dots \frac{n}{2}$,

$$\|Qx_i\| \leq \sqrt{2}dn^{\frac{1}{p}-\frac{1}{2}}.$$

Since $\left(\sum_{i=1}^{n/2} \|Tx_i\|^2\right)^{\frac{1}{2}} \leq \pi_2(T) \leq \epsilon_0 M^{\frac{p}{4}}\sqrt{n}$, we can also assume without loss of generality that $\|Tx_1\| \leq \sqrt{2}\epsilon_0 M^{\frac{p}{4}}$. But then

$$\begin{aligned} \sqrt{2}dn^{\frac{1}{p}-\frac{1}{2}} \quad &\geq \quad \|Qx_1\| \quad = \inf_{h\in H} \|x_1 - h\| \\ &\geq \quad \frac{\|(I-T)x_1\|}{\|I-T\|} \quad \geq \quad \frac{1 - \|Tx_1\|}{\|I-T\|} \\ &\geq \quad \frac{1 - \sqrt{2}\epsilon_0 M^{\frac{p}{4}}}{\|I-T\|}\ ; \end{aligned}$$

that is,

$$d \geq \frac{1 - \sqrt{2}\epsilon_0 M^{\frac{p}{4}}}{\sqrt{2}\|I-T\|} n^{\frac{1}{2}-\frac{1}{p}}.$$

For sufficiently small γ (e.g., $\gamma \leq \frac{1}{2}(K+1)^{-1}$), this gives a lower bound on ϵ_0 since $d \leq \gamma n^{\frac{1}{2}-\frac{1}{p}}$. $\square$

The "consequently" statement follows by duality from Lemma 1 in [148]. The trick of Mascioni's [153] mentioned at the beginning of section 2 can be used to answer a question Pe$czyskiaskedtheauthorstwelvey$ well-embeds into $l_\infty^{(1+\epsilon)n}$ for each $\epsilon > 0$.

Since l_∞^{2n} has an n-dimensional quotient F with d_F bounded independently of n by Kašin's theorem ([161] or [150], corollary 6.4), Proposition 3.2 gives a strong negative answer to Pe$czyski'squestion.$

Proposition 3.9.10. *Set $G = l_\infty^n$, let H be a subspace of G, set $d = d_{G/H}$, and assume that H is K-isomorphic to a subspace of l_∞^k. Then $d \geq e^{-2}(K+1)^{-1}\big(\frac{n-k}{Log k}\big)^{\frac{1}{2}}$.*

Proof. Let $u\colon H \to l_\infty^k$ satisfy $\|u\| = 1$, $\|u^{-1}_{|uH}\| \leq K$, let U be a norm one extension of u to an operator from G to l_∞^k, let S be an extension of u^{-1} to an operator from l_∞^k to G with $\|S\| \leq K$, and set $T = SU$. So $T_{|H} = I_H$ and $\|T\| \leq K$. Let $Q\colon G \to G/H$ be the quotient map and set $E = \ker(T)$, so that $\dim E \geq n - k$. Thus (see the argument at the beginning of section 2)

$$d_E \leq \|I - T\| d \leq (K+1)d\,.$$

But by [162, p.182](let $s = Logk$ there),

$$d_E \geq e^{-2}\Big(\frac{n-k}{Logk}\Big)^{\frac{1}{2}}. \qquad \square$$

$\square$

We conclude with two open problems related to the material in section 1. **Problem 3.3.** If X is a weak Hilbert space, then is $k_X(K,n)$ proportional to n for all $K > 1$? Since weak Hilbert spaces are superreflexive, for all $K > 1$, $k_X(K,n) < \infty$ for every weak Hilbert space X by a result of Lindenstrauss and Tzafriri [155], but their argument does not give a good estimate of $k_X(K,n)$ for K close to one when one has a good estimate for large K.

For the known weak Hilbert spaces X, the growth rate of $k_X(K,n) - n$ is very slow (cf.[145, 163]), at least for sufficiently large K. It follows from recent work of Nielsen and Tomczak-Jaegermann that $k_X(K,n) - n$ has the same kind of slow growth for any weak Hilbert space X which has an unconditional basis. On the other hand, we do not know any improvement of the result presented in section 2 for general weak Hilbert spaces. This suggests: **Problem 3.4.** *If X is a weak Hilbert space, does there exist K so that $k_X(K,n) - n = o(n)$?*

3.10 Sub-Riemannian metrics for quantum Heisenberg manifolds

Nik Weaver

Every Heisenberg manifold carries a natural "sub-Riemannian" metric with interesting properties. We describe the corresponding noncommutative metric structure for Rieffel's quantum Heisenberg manifolds [164]. The purpose of this paper is to study an analog, for quantum Heisenberg manifolds, of the natural sub-Riemannian metrics on classical Heisenberg manifolds. Quantum Heisenberg manifolds were defined in[164] and they have been further investigated in[165–168]. They are interesting for several reasons, one being just because they are tractable examples of noncommutative manifolds. This means that, like the related but simpler noncommutative tori, quantum Heisenberg manifolds provide a nice concrete setting in which to explore noncommutative geometry.

Our treatment of noncommutative metrics is based on Connes' approach [171]. But we prefer to work with abstract derivations rather than the concretely presented derivations implicit in Connes' unbounded Fredholm modules. See[172] for further discussion. Noncommutative metric structure usually arises via an analog of the classical exterior derivative d on a Riemannian manifold. Classically, this map may be realized as a derivation from $\mathrm{Lip}(X) \subset L^\infty(X)$ into the module of bounded measurable 1-forms. The graph of this derivation is weak*-closed, a property which is characteristic of the domain being a Lipschitz algebra[172]. In some sense the differentiable structure resides in the map d, while the metric structure resides in its domain $\mathrm{Lip}(X)$. There are some examples where one has the latter sort of structure but not the former ([173, 174])). An interesting feature of the present work is that from the noncommutative or algebraic point of view, sub-Riemannian metrics are very close to genuine Riemannian metrics. In the language of the previous paragraph, the exterior derivative corresponding to a sub-Riemannian metric is given by composing d with orthogonal projection onto a closed submodule of $\Omega^1(X)$.

This work followed a suggestion by Marc Rieffel, and was helped by discussions about the Heisenberg group with Daniel Allcock. We adopt the following notational conventions: c is a fixed positive integer; $\hbar$, μ, and ν are fixed real numbers; and H is the Hilbert space $L^2(\mathbf{R} \times \mathbf{T} \times \mathbf{Z})$.

3.10.1 Sub-Riemannian structure for classical Heisenberg manifolds

Let M be a connected Riemannian manifold. It is well-known that M has a natural metric such that the distance between two points x and y satisfies

$$d(x, y) = \inf\{l(p) : p \text{ is a path from } x \text{ to } y\},$$

where $l(p)$ denotes the length of p.

Now let B be a subbundle of the tangent bundle TM. We can use it to define a new metric d_B by setting

$$d_B(x, y) = \inf\{l(p) : p \text{ is a path from } x \text{ to } y \\ \text{which is everywhere tangent to } B\}.$$

This is a *sub-Riemannian* or *Carnot-Carathéodory* metric. A good general reference on this topic is [170]. Note that we must either require that any two points can be connected by a path which is tangent to B, or else allow distances to be infinite.

The simplest non-trivial example of a sub-Riemannian metric arises on the Heisenberg group. This example is discussed at length in[175]. We now give a brief account of the corresponding construction for Heisenberg manifolds.

The *(continuous) Heisenberg group* G is the set of all real 3×3 matrices of the form

$$\begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}$$

with product inherited from the matrix ring $M_3(\mathbb{R})$. For any positive integer c the set H_c of elements for which x, y, and cz are integers constitutes a discrete subgroup of G, and the quotient construction yields the *Heisenberg manifold* $M_c = G/H_c$. G acts on M_c from the left. G can be identified with $\mathbb{R}^3$ and so it carries a natural differentiable manifold structure. However, the Euclidean metric on $\mathbb{R}^3$ is not compatible with the group structure of G. Instead, we give G the unique right-invariant Riemannian metric which agrees with the Euclidean metric at the origin. Concretely, the three vectors

$$\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \qquad \frac{\partial}{\partial z}$$

define an orthonormal basis at each point $(x, y, z) \in G$.

Since this Riemannian metric is right-invariant it descends to M_c. The span of the two vector fields $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ is then a bundle B of tangent planes over M_c. (In fact this is a contact subbundle of TM_c, the kernel of the contact 1-form $\eta = dz - xdy$.) We use it to give M_c a sub-Riemannian metric d_B by the procedure described above.

Interestingly, this metric is finite. That is, even though M_c is three-dimensional any two points can be joined by a path whose tangent vector at each point is in the span of $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ [176].

Recall that G acts on M_c from the left. The vector fields $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ can be recovered from this action. To see this consider the two one-parameter subgroups of G of the form $x = r$, $y = z = 0$ and $y = s$, $x = z = 0$; the generators of their actions on M_c are the two desired vector fields. That is, flowing along the vector fields $\partial/\partial x$ and $\partial/\partial y + x(\partial/\partial z)$ produces the actions α_r and β_s on M_c defined by

$$\alpha_r(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} \cdot A, \qquad \beta_s(A) = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot A$$

$(A \in M_c)$. We will use this observation to define an analogous construction for quantum Heisenberg manifolds.

3.10.2 Quantum Heisenberg manifolds

We recall the definition of the quantum Heisenberg C*-algebras, and define corresponding von Neumann algebras. Recall that c is a fixed positive integer, $\hbar$, μ, and ν are fixed real numbers, and $H = L^2(\mathbf{R} \times \mathbf{T} \times \mathbf{Z})$.

Definition 4 (Theorem 5.5)**.** *[164] Let S^c denote the space of C^∞ functions Φ on $\boldsymbol{R} \times \boldsymbol{T} \times \boldsymbol{Z}$ which satisfy*

(a) $\Phi(x+k, y, p) = e^{ickpy}\Phi(x, y, p)$ for all $k \in 2\pi\mathbb{Z}$; and

(b) for every polynomial P on $\mathbb{Z}$ and every partial differential operator $\tilde{X} = \partial^{m+n}/\partial x^m \partial y^n$ on $\boldsymbol{R} \times \boldsymbol{T}$ the function $P(p) \cdot (\tilde{X}\Phi)(x, y, p)$ is bounded on $C \times \mathbb{Z}$ for any compact subset C of $\boldsymbol{R} \times \boldsymbol{T}$.

Define an action of $\Phi \in S^c$ on H by

$$(\Phi\xi)(x, y, p) = \sum_q \Phi(x - \hbar(q-2p)\mu, y - \hbar(q-2p)\nu, q)\xi(x, y, p-q)$$

(recall that $H = L^2(\mathbf{R} \times \mathbf{T} \times \mathbf{Z})$. Then let $D_\hbar = D^{\mu,\nu}_{\hbar,c}$ be the norm closure of $S^c \subset B(H)$ and let $N_\hbar = N^{\mu,\nu}_{\hbar,c}$ be its weak operator closure. It is shown in[164] that $D_\hbar$ is a C*-algebra, and it follows that $N_\hbar$ is a von Neumann algebra. Note that our conventions differ from [164] by a factor of 2π in the x variable. The C*-algebras $D_\hbar$ are classified up to isomorphism in [166, 167]

We require alternative characterizations of $D_\hbar$ and $N_\hbar$. The results are analogous to, but a bit more complicated than, corresponding facts about noncommutative tori[172]. (Another characterization of $D_\hbar$ is given in[168].) Our main tool is a kind of Fourier expansion of elements of $N_\hbar$, given in the next definition. We record its basic properties in the subsequent lemmas.

Definition 5. *For $t \in \mathbb{R}$ and $n \in \mathbb{Z}$ define unitary operators U_t and Y_n on H by*

$$(U_t\xi)(x, y, p) = e^{ipt}\xi(x, y, p) \qquad \text{and} \qquad (Y_n\xi)(x, y, p) = \xi(x, y, p+n).$$

For any $T \in B(H)$ and $n \in \mathbb{Z}$ define $a_n(T) \in B(H)$ by

$$a_n(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U_t T U_t^{-1} e^{-int} dt.$$

This and all other operator integrals are taken in the weak operator sense, i.e.

$$\langle a_n(T)\xi, \eta\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle U_t T U_t^{-1}\xi, \eta\rangle e^{-int} dt$$

for all $\xi, \eta \in H$. We regard $a_n(T)$ as a sort of Fourier coefficient of T; similarly, for $N \in \mathbb{N}$ we define the Cesaro mean $\sigma_N(T)$ by

$$\sigma_N(T) = \sum_{-N}^{N}\Big(1 - \frac{|n|}{N+1}\Big)a_n(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U_t T U_t^{-1} K_N(t) dt,$$

where K_N is the Fejér kernel

$$K_N(t) = \sum_{n=-N}^{N}\Big(1 - \frac{|n|}{N+1}\Big)e^{-int} = \frac{1}{N+1}\Big(\frac{\sin((N+1)t/2)}{\sin(t/2)}\Big)^2.$$

Lemma 3.10.1. *For any $T \in B(H)$ we have $\sigma_N(T) \to T$ weak operator as $N \to \infty$. If the map $t \mapsto U_t T U_t^{-1}$ is continuous for the norm topology on $B(H)$ then $\sigma_N(T) \to T$ in norm as $N \to \infty$.*

Proof. For the weak operator statement, pick $\xi, \eta \in H$ and observe that the map

$$t \mapsto \langle (U_t T U_t^{-1} - T)\xi, \eta\rangle$$

is continuous and vanishes at $t = 0$. Therefore its integral against K_N goes to zero as $N \to \infty$ (e.g. see [11]), hence

$$\langle (\sigma_N(T) - T)\xi, \eta\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \langle (U_t^{-1} T U_t) - T\xi, \eta\rangle K_N(t)dt \to 0.$$

This shows that $\sigma_N(T) \to T$ weak operator.

For the norm statement, note that the function $t \mapsto \|U_t^{-1} T U_t^{-1} - T\|$ is continuous and vanishes at zero. Therefore

$$\begin{aligned}\|\sigma_N(T) - T\| &= \frac{1}{2\pi}\Big\| \int_{-\pi}^{\pi} (U_t T U_t^{-1} - T) K_N(t)dt\Big\| \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \|U_t T U_t^{-1} - T\| K_N(t)dt \to 0\end{aligned}$$

as $N \to \infty$. ☐

Lemma 3.10.2. *For any $T \in B(H)$, the operator $Y_n a_n(T)$ preserves constant p subspaces of $H = L^2(\boldsymbol{R} \times \boldsymbol{T} \times \boldsymbol{Z})$ ($p \in \mathbb{Z}$).*

Proof. Observe that $Y_n a_n(T)$ commutes with U_s for all s:

$$\begin{aligned}Y_n a_n(T) U_s &= \frac{1}{2\pi} Y_n \int_{-\pi}^{\pi} U_t T U_t^{-1} U_s e^{-int} dt \\ &= \frac{1}{2\pi} Y_n U_s \int_{-\pi}^{\pi} U_{t-s} T U_{t-s}^{-1} e^{-int} dt \\ &= \frac{1}{2\pi} e^{ins} U_s Y_n \int_{-\pi}^{\pi} U_t T U_t^{-1} t e^{-in(t+s)} dt \\ &= U_s Y_n a_n(T).\end{aligned}$$

But the operators U_s generate the von Neumann algebra $l^\infty(\mathbb{Z}) \subset B(H)$, so we conclude that $Y_n a_n(T)$ preserves the constant p subspaces of H. ☐

Lemma 3.10.3. *Let $T \in B(H)$. Suppose T commutes with the operators V_f, W_k and X_r defined by*

$$\begin{aligned}(V_f\xi)(x, y, p) &= f(x, y)\xi(x, y, p) \\ (W_k\xi)(x, y, p) &= e^{-ick(p^2\hbar\nu + py)}\xi(x + k, y, p) \\ (X_r\xi)(x, y, p) &= \xi(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r)\end{aligned}$$

for all $f \in L^\infty(\boldsymbol{R} \times \boldsymbol{T})$, $k \in 2\pi\mathbb{Z}$, and $r \in \mathbb{Z}$. Then $a_n(T)$ satisfies

$$(a_n(T)\xi)(x, y, p) = g(x, y, p)\xi(x, y, p - n)$$

for some $g \in L^\infty(\boldsymbol{R} \times \boldsymbol{T} \times \boldsymbol{Z})$*, and the function* g *satisfies*

$$g(x+k, y, p) = e^{-ick((n^2-2np)\hbar\nu - ny)} g(x, y, p) \tag{$*$}$$

($k \in 2\pi\mathbb{Z}$*) and*

$$g(x, y, p) = g(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r) \tag{$\dagger$}$$

($r \in \mathbb{Z}$*).*

The function $\Phi \in L^\infty(\boldsymbol{R} \times \boldsymbol{T} \times \boldsymbol{Z})$ *defined by* $\Phi(x, y, p) = 0$ *for* $p \neq n$ *and*

$$\Phi(x, y, n) = g(x - \hbar n\mu, y - \hbar n\nu, n)$$

satisfies condition (a) of Definition 1, and $a_n(T) = \Phi$ *where* Φ *acts on* H *as in Definition 1.*

Proof. Every V_f commutes with T by hypothesis and with both U_t and Y_n by easy computations. It follows that V_f also commutes with $Y_n a_n(T)$. Thus $Y_n a_n(T)$, which preserves constant p subspaces by Lemma 4, must be a multiplication operator on each constant p subspace of H. So $a_n(T)$ has the form

$$(a_n(T)\xi)(x, y, p) = g(x, y, p)\xi(x, y, p - n)$$

for some $g \in L^\infty(\mathbf{R} \times \mathbf{T} \times \mathbf{Z})$. Next observe that every W_k commutes with both T and U_t, hence W_k commutes with $a_n(T)$. So

$$(a_n(T)W_k\xi)(x, y, p) = g(x, y, p)e^{-ick((p-n)^2\hbar\nu + (p-n)y)}\xi(x + k, y, p - n)$$

equals

$$(W_k a_n(T)\xi)(x, y, p) = e^{-ick(p^2\hbar\nu + py)} g(x + k, y, p)\xi(x + k, y, p - n),$$

which implies $(*)$.

Similarly, we have $X_r U_t = e^{irt} U_t X_r$, hence X_r commutes with $U_t T U_t^{-1}$ and therefore with $a_n(T)$, which implies that

$$(a_n(T)X_r\xi)(x, y, p) = g(x, y, p)\xi(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r - n)$$

equals

$$(X_r a_n(T)\xi)(x, y, p) = g(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r)\xi(x - 2\hbar r\mu, y - 2\hbar r\nu, p + r - n).$$

From this we obtain $(\dagger)$.

Finally, define Φ as in the statement of the lemma. It follows more or less immediately from $(*)$ that Φ satisfies condition (a) of Definition 1. Furthermore it follows from $(\dagger)$ that the action of Φ on H given in Definition 1 agrees with the action of $a_n(T)$, that is, taking $r = n - p$,

$$\begin{aligned}
(a_n(T)\xi)(x, y, p) &= g(x, y, p)\xi(x, y, p - n) \\
&= g(x - 2\hbar(n - p)\mu, y - 2\hbar(n - p)\nu, n)\xi(x, y, p - n) \\
&= \Phi(x - \hbar(n - 2p)\mu, y - \hbar(n - 2p)\nu, n)\xi(x, y, p - n) \\
&= \sum_q \Phi(x - \hbar(q - 2p)\mu, y - \hbar(q - 2p)\nu, q)\xi(x, y, p - q).
\end{aligned}$$

(Note that even if Φ does not satisfy condition (b) of Definition 1, it still acts as a bounded operator on H. In fact $Y_n\Phi$ is a multiplication operator and so the operator norm of Φ equals $\|\Phi\|_\infty$.) □

Theorem 3.10.4. *Let $T \in B(H)$. Then $T \in N_\hbar$ if and only if T commutes with the operators V_f, W_k, and X_r defined in Lemma 5 for all $f \in L^\infty(\boldsymbol{R} \times \boldsymbol{T})$, $k \in 2\pi\mathbb{Z}$, and $r \in \mathbb{Z}$.*

Proof. The forward direction can be demonstrated by checking that every $\Phi \in S^c \subset B(H)$ commutes with V_f, W_k, and X_r. This is an elementary calculation and we omit it. Suppose T commutes with V_f, W_k, and X_r; we must show that $T \in N_\hbar$. Since $\sigma_N(T) \to T$ weak operator by Lemma 3, it will suffice to show $\sigma_N(T) \in N_\hbar$ for all $N \in \mathbb{N}$; and since $\sigma_N(T)$ is a linear combination of the Fourier coefficients $a_n(T)$ it will suffice to show $a_n(T) \in N_\hbar$ for all $n \in \mathbb{N}$.

Let g and Φ be the functions defined in Lemma 5 and let (h_m) be a sequence of functions in $C^\infty(\mathbf{R} \times \mathbf{T})$ with $\mathrm{supp}(h_m) \subset [-1/m, 1/m] \times [-1/m, 1/m]$ and $\|h_m\|_1 = 1$. Define smoothings Φ_m of Φ by twisted convolution:

$$\Phi_m(x, y, n) = \int_{\mathbf{R}\times\mathbf{T}} h_m(r, s)\Phi(x - r, y - s, n)e^{icnxs}\,drds.$$

Then Φ_m satisfies condition (a) of Definition 1 because

$$\begin{aligned}\Phi_m(x + k, y, n) &= \int h_m(r, s)\Phi(x + k - r, y - s, n)e^{icn(x+k)s}\,drds\\ &= \int h_m(r, s)e^{ickn(y-s)}\Phi(x - r, y - s, n)e^{icn(x+k)s}\,drds\\ &= e^{ickny}\Phi_m(x, y, n);\end{aligned}$$

and Φ_m satisfies condition (b) of Definition 1 because it is C^∞ and supported on $p = n$. So $\Phi_m \in S^c$, and a standard application of Fubini's theorem and dominated convergence shows that $\Phi_m \to \Phi$ weak* in $L^\infty(\mathbf{R} \times \mathbf{T} \times \mathbf{Z})$. Thus the multiplication operators $Y_n\Phi_m$ converge weak operator to $Y_n\Phi$, so that $\Phi_m \to \Phi$ weak operator. This shows that $a_n(T) = \Phi$ belongs to $N_\hbar$. ☐

The corresponding characterization of the C*-algebra $D_\hbar$ can be stated most naturally in terms of the action of the Heisenberg group G on the $D_\hbar$ given in [164].

Definition 6. *[164, p.557] For $r, s, t \in \mathbb{R}$ let $U_{(r,s,t)}$ be the unitary operator on H defined by*

$$(U_{(r,s,t)}\xi)(x, y, p) = e^{ip(t+cs(x+\hbar p\mu-r))}\xi(x - r, y - s, p).$$

Then $L_{(r,s,t)}(T) = U_{(r,s,t)}TU^{-1}_{(r,s,t)}$ defines an action L of G on $B(H)$, where we take $(r, s, t) \in \mathbb{R}^3 \cong G$. A short computation shows that this action preserves S^c, hence it preserves $D_\hbar$ and $N_\hbar$.

By specializing to the three coordinate axes in $G \cong \mathbb{R}^3$ we get three actions α, β, γ of $\mathbb{R}$, defined by

$$\begin{aligned}\alpha_r(T) &= U_{(r,0,0)}TU^*_{(r,0,0)}\\ \beta_s(T) &= U_{(0,s,0)}TU^*_{(0,s,0)}\\ \gamma_t(T) &= U_{(0,0,t)}TU^*_{(0,0,t)}.\end{aligned}$$

(Note that $U_{(0,0,t)}$ equals the unitary U_t of Definition 2. Thus, for instance, $a_n(T) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\gamma_t(T)e^{-int}dt$.)

For $\Phi \in S^c \subset B(H)$ we have

$$\begin{aligned}\alpha_r(\Phi)(x,y,p) &= \Phi(x-r,y,p)\\ \beta_s(\Phi)(x,y,p) &= e^{ipscx}\Phi(x,y-s,p)\\ \gamma_t(\Phi)(x,y,p) &= e^{ipt}\Phi(x,y,p).\end{aligned}$$

Lemma 3.10.5. *Let $\Phi \in S^c$. Then there exists $K > 0$ such that*

$$\|\alpha_r(\Phi) - \Phi\| \leq Kr \qquad \text{and} \qquad \|\beta_s(\Phi) - \Phi\| \leq Ks$$

for all $r, s > 0$.

Proof. By condition (b) of Definition 1 there exists a positive function $f \in L^1(\mathbb{Z})$ such that

$$\Big|\frac{\partial \Phi}{\partial x}(x,y,p)\Big| \leq f(p)$$

for all $p \in \mathbb{Z}$. So for $r > 0$ we have

$$|(\alpha_r(\Phi) - \Phi)(x,y,p)| = |\Phi(x-r,y,p) - \Phi(x,y,p)| \leq r \cdot f(p).$$

It follows that $\|\alpha_r(\Phi) - \Phi\| \leq r \cdot \|f\|_1$.

Now use condition (b) of Definition 1 to find a positive function f_1 such that $pf_1(p) \in L^1(\mathbb{Z})$ and $|\Phi(x,y,p)| \leq f_1(p)$ for all $p \in \mathbb{Z}$ and a positive function $f_2 \in L^1(\mathbb{Z})$ such that

$$\Big|\frac{\partial \Phi}{\partial y}(x,y,p)\Big| \leq f_2(p)$$

for all $p \in \mathbb{Z}$. Also note that

$$\sup_{0 \leq x < 2\pi} |e^{ipcsx} - 1| \leq 2\pi cps.$$

Then

$$\begin{aligned}|(\beta_s(\Phi) - \Phi)(x,y,p)| &= |e^{ipcsx}\Phi(x,y-s,p) - \Phi(x,y,p)|\\ &\leq |e^{ipcsx} - 1| f_1(p) + |\Phi(x,y-s,p) - \Phi(x,y,p)|\\ &\leq (2\pi cpf_1(p) + f_2(p))s\end{aligned}$$

for all $x \in [0,2\pi)$. But for any $x \in [0,2\pi)$ and $k \in 2\pi\mathbb{Z}$ we have

$$\begin{aligned}(\beta_s(\Phi) - \Phi)(x+k,y,p) &= e^{ipcs(x+k)}\Phi(x+k,y-s,p) - \Phi(x+k,y,p)\\ &= e^{ipcs(x+k)}e^{ickp(y-s)}\Phi(x,y-s,p) - e^{ickpy}\Phi(x,y,p)\\ &= e^{icpky}(\beta_s(\Phi) - \Phi)(x,y,p),\end{aligned}$$

so that the previous estimate holds for all $x \in \mathbb{R}$. It follows that $\|\beta_s(\Phi) - \Phi\| \leq (2\pi c\|pf_1\|_1 + \|f_2\|_1)s$. ☐

Theorem 3.10.6. *Let $T \in N_\hbar$. Then $T \in D_\hbar$ if and only if the maps $r \mapsto \alpha_r(T)$ and $s \mapsto \beta_s(T)$ are continuous for the norm topology on $N_\hbar$.*

Proof. ($\Rightarrow$) The set of operators T for which α and β are norm-continuous is easily seen to be norm-closed. Thus it suffices to show that every $\Phi \in S^c \subset B(H)$ has this property. This was shown in Lemma 8.

($\Leftarrow$) Let $T \in N_\hbar$ and suppose α and β are norm-continuous for T. It follows that γ is also norm-continuous for T by the identity

$$\gamma_t = \beta_{t'}^{-1}\alpha_{t'}^{-1}\beta_{t'}\alpha_{t'}$$

where $t' = \sqrt{t/c}$. Thus $\sigma_N(T) \to T$ in norm as $N \to \infty$ by Lemma 3. Therefore, to prove that $T \in D_\hbar$ it will suffice to show that $\sigma_N(T) \in D_\hbar$, or indeed that $a_n(T) \in D_\hbar$. Now α and γ commute, so

$$\begin{aligned}
\|\alpha_r(a_n(T)) - a_n(T)\| &= \frac{1}{2\pi}\|\int_{-\pi}^{\pi}(\alpha_r(\gamma_t(T)) - \gamma_t(T))e^{-int}dt\| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\|\alpha_r(\gamma_t(T)) - \gamma_t(T)\|dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\|\alpha_r(T) - T\|dt \\
&= \|\alpha_r(T) - T\|.
\end{aligned}$$

Since T is norm-continuous for α, this shows that $a_n(T)$ is as well; the same argument shows that $a_n(T)$ is norm-continuous for β.

Using the fact that $\alpha_r(Y_n a_n(T)) = Y_n\alpha_r(a_n(T))$ we get that $\alpha_r(Y_n a_n(T))$ is continuous in norm as a function of r. Similarly, a short computation shows that $\beta_s(Y_n)$ is continuous in norm as a function of s, hence $\beta_s(Y_n a_n(T)) = \beta_s(Y_n)\beta_s(a_n(T))$ is also continuous in norm. It is then standard that the functions g and Φ defined in Lemma 5 must be uniformly continuous. We let Φ act on H by the formula given in Definition 1, so that $a_n(T) = \Phi$ as operators.

Now, just as in the proof of Theorem 6, we can smooth Φ by taking a twisted convolution with a C^∞ approximate unit of L^1-norm one, to get a sequence (Φ_m) in S^c. But since Φ is continuous, $\Phi_m \to \Phi$ in sup norm = operator norm, hence $a_n(T) = \Phi \in D_\hbar$. ☐

Corollary 3.10.7. *$D_\hbar$ consists of precisely the elements of $N_\hbar$ for which the action of G is norm-continuous.*

Proof. Norm-continuity for G implies norm-continuity for α and β, so one direction follows immediately from Theorem 9. For the other direction, we also know from Theorem 9 that every element of $D_\hbar$ is norm-continuous for α and β, and that this implies norm-continuity for γ as well. But α, β, and γ generate G, so this is enough. ☐

3.10.3 The noncommutative sub-Riemannian metric

There is a natural noncommutative sub-Riemannian metric on $N_\hbar$, and it can be presented in both local and global forms. The local version is a sort of noncommutative exterior derivative, while the global version is the noncommutative Lipschitz algebra that is the former's domain.

Definition 7. *Let $E = N_\hbar \oplus N_\hbar$; we regard it as a Hilbert bimodule over $N_\hbar$ with left and right $N_\hbar$-valued inner products given by*

$$\langle x_1 \oplus x_2, y_1 \oplus y_2\rangle_l = x_1y_1^* + x_2y_2^*$$

and

$$\angle x_1 \oplus x_2, y_1 \oplus y_2\rangle_r = x_1^* y_1 + x_2^* y_2.$$

Let δ_1 and δ_2 be the generators of the actions α and β defined in the last section, i.e.

$$\delta_1(x) = \lim_{r\to 0} \frac{\alpha_r(x) - x}{r} \qquad \text{and} \qquad \delta_2(x) = \lim_{s\to 0} \frac{\beta_s(x) - x}{s}$$

for all $x \in N_\hbar$ for which the limits exist in the weak operator sense. Define ${}_\hbar = \mathrm{dom}(\delta_1) \cap \mathrm{dom}(\delta_2)$ and define $d : {}_\hbar \to E$ by $d(x) = \delta_1(x) \oplus \delta_2(x)$. Give ${}_\hbar$ the norm

$$\|x\|_L = \max(\|x\|, \|d(x)\|_l, \|d(x)\|_r)$$

where $\|\cdot\|_l$ and $\|\cdot\|_r$ are the left and right Hilbert module norms on E.

In the case $\hbar = 0$, δ_1 and δ_2 are genuine partial derivatives, and $d(x)$ is the projection of the exterior derivative of x onto the cotangent subbundle dual to B.

The following alternative characterization of ${}_\hbar$ is useful. It follows immediately from ([177], Proposition 3.1.6).

Lemma 3.10.8. *Let $x \in N_\hbar$. Then $x \in {}_\hbar$ if and only if* $\sup_{r>0} \|\alpha_r(x) - x\|/r$ *and* $\sup_{s>0} \|\beta_s(x) - x\|/s$ *are finite.*

Theorem 3.10.9. *The map d and its domain ${}_\hbar$ have the following properties:*

(a). d is an unbounded derivation with weak-closed graph.*

(b). ${}_\hbar$ is a dual Banach algebra. It contains S^c and is densely contained in $D_\hbar$.

(c). If $\hbar = 0$ then ${}_\hbar$ is naturally identified with the algebra of functions on M_c which are Lipschitz for the sub-Riemannian metric defined in section 1, and $\|d(x)\|_l = \|d(x)\|_r$ equals the Lipschitz number of x, for any $x \in {}_0 \cong \mathrm{Lip}(M_c)$.

Proof. (a). The fact that d is a derivation, i.e. is linear and satisfies the Liebnitz formula $d(xy) = xd(y) + d(x)y$, is an elementary calculation. Weak*-closure of the graph follows from ([7], Proposition 3.1.6).

(b). The norm $\|\cdot\|_L$ equals the graph norm when ${}_\hbar$ is identified with the graph of d by the map $x \mapsto x \oplus d(x)$ and E is given the max of its left and right Hilbert norms (which is equivalent to the von Neumann algebra norm on E). Thus ${}_\hbar$ is isometric to a weak*-closed subspace of a dual Banach space, hence ${}_\hbar$ is a dual space. It is an algebra because $\mathrm{dom}(\delta_1)$ and $\mathrm{dom}(\delta_2)$ are (being domains of derivations).

$S^c \subset {}_\hbar$ follows from Lemma 8 and Lemma 12. ${}_\hbar \subset D_\hbar$ follows from Theorem 9 and Lemma 12, using the fact that continuity of an $\mathbb{R}$-action is equivalent to continuity at 0. Also, ${}_\hbar$ is dense in $D_\hbar$ because it contains S^c.

(c). Let $\hbar = 0$. It is straightforward to check that D_0 and N_0 are, respectively, naturally isomorphic to $C(M_c)$ and $L^\infty(M_c)$; this simply involves taking the Fourier transform in the p variable. Now if $f \in L^\infty(M_c)$ is Lipschitz for the sub-Riemannian metric then it satisfies

$$\|f \circ \phi - f\|_\infty \leq L(f) \cdot r$$

for any isometry ϕ of M_c such that $d_B(\phi(\rho), \rho) = r$ for all $\rho \in M_c$, where $L(f)$ is the Lipschitz number of f. Taking

$$\phi(r, s, t) = (r - h\cos\theta, s - h\sin\theta, t - hr\sin\theta),$$

this shows that $\mathrm{Lip}(M_c) \subset {}_0$ by Lemma 12; and

$$\|d(f)\|_l(\rho) = \operatorname*{ess\,sup}_{\theta} |\delta_1(f)(p)|^2\cos^2\theta + |\delta_2(f)(p)|^2\sin^2\theta \le L(f)$$

for almost every $\rho \in M_c$, so $\|d(f)\| \le L(f)$.

Conversely, let f be any function in ${}_0$ and let $\rho, \sigma \in M_c$. By ([169], Theorem 2.7) there exists a constant velocity geodesic $p : [0, 1] \to M_c$ which is everywhere tangent to the subbundle B of TM_c defined in section 1 and which satisfies $p(0) = \rho$, $p(1) = \sigma$, and $l(p) = d_B(\rho, \sigma)$. Then the function $g = f \circ p : [0, 1] \to \mathbb{C}$ satisfies $g(0) = f(\rho)$, $g(1) = f(\sigma)$, and $g'(t) = \langle d(f)(t), dp(t)\rangle$ for almost every $t \in [0, 1]$. This implies that g is Lipschitz with

$$L(g) \le \|dp\| = \|d(f)\| \cdot l(p),$$

so

$$\frac{|f(\rho) - f(\sigma)|}{d_B(\rho, \sigma)} = \frac{g(0) - g(1)}{l(p)} \le L(g)/l(p) \le \|d(f)\|.$$

Taking the supremum over all ρ and σ shows that f is Lipschitz and $L(f) \le \|d(f)\|$. □

We conclude this section with a proof that the unit ball of ${}_\hbar$ is compact in operator norm. In the commutative case, this is true of $\mathrm{Lip}(X)$ precisely when X is compact. In addition it was proved for noncommutative Lipschitz algebras associated to noncommutative tori in [172, 173], and our proof here uses basically the same method. These results also follow from an unpublished theorem of Rieffel which deals with the general situation of a Lie group acting on a Banach space . I do not know whether that line of reasoning implies our current result (it is not obvious because $\mathbb{R}^2$, the Lie group that appears here, is not compact).

Lemma 3.10.10. *For any $\epsilon > 0$ there exists N large enough that $\|x - \sigma_n(x)\| \le \epsilon$ for all $x \in \mathrm{ball}({}_\hbar)$ and $n \ge N$.*

Proof. Recall that $\gamma_t = \beta_{t'}^{-1}\alpha_{t'}^{-1}\beta_{t'}\alpha_{t'}$ where $t' = \sqrt{t/c}$. Now for any $y \in \mathrm{ball}({}_\hbar)$ we have

$$\|\alpha_{t'}(y) - y\|, \|\beta_{t'}(y) - y\| \le t' \cdot \|d(y)\| \le t',$$

so that $\|\gamma_t(x) - x)\| \le 4t'$ for any $x \in \mathrm{ball}({}_\hbar)$. So we have

$$\|x - \sigma_n(x)\| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x - \gamma_t(x)\| K_n(t)dt \le \frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sqrt{t/c}K_n(t)dt,$$

and the last formula goes to zero as $n \to \infty$. This is what we needed to show. □

Theorem 3.10.11. *The unit ball of ${}_\hbar$ is compact in operator norm.*

Proof. Let (x_k) be any sequence in $\mathrm{ball}({}_\hbar)$; we will find a convergent subsequence. As in Lemma 5, $Y_n a_n(x_k)$ is multiplication by some function $g_n^k \in L^\infty(\mathbf{R} \times \mathbf{T} \times \mathbf{Z})$. Now

$$\|\alpha_r(Y_n a_n(x_k)) - Y_n a_n(x_k)\| = \|\alpha_r(a_n(x_k)) - a_n(x_k)\|$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \|(\alpha_r(\gamma_t(x_k)) - \gamma_t(x_k))e^{-int}\|dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \|\alpha_r(x_k) - x_k\|dt \\
&\leq r\|\delta_1(x_k)\| \leq r.
\end{aligned}
$$

Similarly $\|\beta_s(Y_n a_n(x_k)) - Y_n a_n(x_k)\| \leq s$, and this implies that the function g_n^k is Lipschitz with Lipschitz number at most 1. Since $[0, 2\pi] \times \mathbf{T} \times \{0\}$ is compact, we may choose a subsequence $g_n^{k_j}$ which converges in sup norm on this set; by $(*)$ and $(\dagger)$ of Lemma 5 this implies that $g_n^{k_j}$ converges in sup norm on all of $\mathbf{R} \times \mathbf{T} \times \mathbf{Z}$. Allowing n to vary, finding successive subsequences for which $g_n^{k_j}$ converges, and diagonalizing, we get a subsequence (x_{j_k}) of (x_k) such that $(g_n^{k_j})$ converges in sup norm for all n.

Let x be an weak operator cluster point of (x_{j_k}) and let $Y_n(a_n(x))$ be multiplication by g_n. Then g_n is a cluster point of $(g_n^{k_j})$, hence $g_n^{k_j} \to g_n$.

Given $\epsilon > 0$, by Lemma 14 we can find a positive integer N such that $\|x - \sigma_N(x)\| \leq \epsilon$ and $\|x_{k_j} - \sigma_N(x_{k_j})\| \leq \epsilon$ for all j. Taking M large enough that $j \geq M$ implies

$$
\|g_n - g_n^{k_j}\|_\infty \leq \epsilon/(2N+1)
$$

for all $|n| \leq N$, we get

$$
\begin{aligned}
\|\sigma_N(x) - \sigma_N(x_{k_j})\| &\leq \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big)\|a_n(x) - a_n(x_{k_j})\| \\
&= \sum_{n=-N}^{N} \Big(1 - \frac{|n|}{N+1}\Big)\|g_n - g_n^{k_j}\| \\
&\leq \epsilon.
\end{aligned}
$$

Thus

$$
\|x - x_{k_j}\| \leq \|x - \sigma_N(x)\| + \|\sigma_N(x) - \sigma_N(x_{k_j})\| + \|\sigma_N(x_{k_j}) - x_{k_j}\| \leq 3\epsilon
$$

for $j \geq M$. So $x_{k_j} \to x$ in operator norm. ☐

3.10.4 Further properties

In this section we first identify the sub-Riemannian noncommutative Lipschitz algebra $_\hbar$ defined in section 3 with a noncommutative Hölder algebra. This generalizes the classical fact (see[175]) that the sub-Riemannian metric on the Heisenberg manifolds is comparable to the square root of the Riemannian metric in the z direction. Then we use work of Sauvageot [178] to establish the existence of a heat semigroup on $N_\hbar$ and identify its generator with a noncommutative Laplacian.

Definition 8. *Let $A, B, C \in (0, 1]$ and define $^{A,B,C}_\hbar$ to be the set of $x \in N_\hbar$ for which there exists a constant $K \geq 0$ such that*

$$
\|x - \alpha_r(x)\| \leq Kr^A, \qquad \|x - \beta_s(x)\| \leq Ks^B, \qquad \|x - \gamma_t(x)\| \leq Kt^C
$$

for all $r, s, t > 0$. Let $L(x) = L^{A,B,C}(x)$ be the smallest possible value of K and norm ${}_{\hbar}^{A,B,C}$ by

$$\|x\|_{A,B,C} = \max(\|x\|, L(x)).$$

Note that $A' \le A, B' \le B, C' \le C$ implies ${}_{\hbar}^{A,B,C} \subset {}_{\hbar}^{A',B',C'}$. Indeed, if $K = \max(L^{A,B,C}(x), 2\|x\|)$ then

$$\|x - \alpha_r(x)\| \le Kr^A \le Kr^{A'} for$$

$r \le 1 2\|x\| \le Kr^{A'} for r \ge 1,$

$$andsimilarly for$$

β and γ, so that $L^{A',B',C'}(x) \le \max(L^{A,B,C}(x), 2\|x\|)$.

Note also that ${}_{\hbar}^{1,1,1} = \mathrm{dom}(\alpha) \cap \mathrm{dom}(\beta) \cap \mathrm{dom}(\gamma)$ by an obvious extension of Lemma 12. For this reason we can realize ${}_{\hbar}^{1,1,1}$ as the domain of a derivation into the Hilbert module $N_\hbar \oplus N_\hbar \oplus N_\hbar$ in the same way that we treated ${}_{\hbar}$ in Definition 11. However, for non-unit values of A, B, or C Hilbert modules are not appropriate. We instead use a construction from [173] to handle this case.

Definition 9. *Let*

$$F = \bigoplus_{t>0}^{\infty} (N_\hbar \oplus N_\hbar \oplus N_\hbar)$$

be the l^∞ direct sum of von Neumann algebras. Regard it as a dual operator $N_\hbar$-bimodule with left action given by the diagonal embedding of $N_\hbar$ in F and right action given by the embedding

$$x \mapsto \bigoplus_{t>0} (\alpha_t(x) \oplus \beta_t(x) \oplus \gamma_t(x)).$$

Define a map $d : {}_{\hbar}^{A,B,C} \to F$ by $d(x) = \bigoplus d_t(x)$ with

$$d_t(x) = \frac{x - \alpha_t(x)}{t^A} \oplus \frac{x - \beta_t(x)}{t^B} \oplus \frac{x - \gamma_t(x)}{t^C}.$$

Theorem 3.10.12. *The map d and its domain ${}_{\hbar}^{A,B,C}$ have the following properties:*

(a). d is an unbounded derivation with weak-closed graph.*

(b). ${}_{\hbar}^{A,B,C}$ is a dual Banach algebra. It is densely contained in $D_\hbar$ and, if $C < 1$, it contains S^c.

(c). The unit ball of ${}_{\hbar}^{A,B,C}$ is compact in operator norm.

Proof. (a). It is routine to check that d is a derivation. To verify weak*-closure of the graph, let $(x_\lambda) \subset {}_{\hbar}^{A,B,C}$ be a bounded net which weak operator converges to $x \in N_\hbar$ and suppose $d(x_\lambda)$ weak operator converges to $\bigoplus y_t^1 \oplus y_t^2 \oplus y_t^3 \in F$. Then restricting attention to the tth summand of F, we have $d_t(x_\lambda) \to y_t^1 \oplus y_t^2 \oplus y_t^3$. Thus

$$(x_\lambda - \alpha_t(x_\lambda))/t^A \to y_t^1$$

(weak operator), but the left side also converges to $(x - \alpha_t(x))/t^A$. An identical argument applies to the other two summands of $d_t(x)$ and so we conclude that $d_t(x_\lambda) \to d_t(x)$. Boundedness of the net then implies that $x \in {}_{\hbar}^{A,B,C}$ and $d(x_\lambda) \to d(x)$.

(b). ${}_{\hbar}^{A,B,C}$ is a dual Banach algebra by the same easy argument used in Theorem 13 (b). It is contained in $D_\hbar$ by Theorem 9. Also, as remarked above it contains $L_\hbar^{1,1,1}$. We claim that ${}_{\hbar}^{1,1,1}$ contains all $\Phi \in S^c$ which are zero for all but a finite number of values of p. This implies density in $D_\hbar$ by Lemma 3.

To prove the claim it suffices to consider only those $\Phi \in S^c$ for which $\Phi(x, y, p) = 0$ unless $p = n$, for some fixed $n \in \mathbf{Z}$. For such Φ the operator norm equals the L^∞ norm, and so

$$\|\gamma_t(\Phi) - \Phi\| = \|(e^{int} - 1)\Phi\|_\infty \leq nt\|\Phi\|_\infty.$$

Together with Lemma 8 this implies $\Phi \in {}_{\hbar}^{1,1,1}$, as claimed. Now suppose $C < 1$ and let Φ be any function in S^c. Choose $N \geq 1 + 2/(1 - C)$. Then by part (b) of Definition 1, there exists a constant K such that $|p^N \Phi(x, y, p)| \leq K$ for $x \in [0, 2\pi]$ and all y and p. By part (a) of Definition 1, this implies that $p^N\Phi$ is bounded on all of $\mathbf{R} \times \mathbf{T} \times \mathbf{Z}$. Now for any $t > 0$ define $\Phi_q(x, y, p) = \delta_{p,q}\Phi(x, y, p)$ and $\Phi' = \Phi - \sum_{|p| \leq t^{-1/(N-1)}+1} \Phi_q$. Then $\|\Phi_q\|_\infty \leq q^{-N}K$ and so we can bound the operator norm of Φ' by

$$\begin{aligned}\|\Phi'\| &\leq \sum_{|q| > t^{-1/(N-1)}+1} q^{-N}K \\ &\leq 2\int_{t^{-1/(N-1)}}^{\infty} q^{-N}K\,dq \\ &= 2\frac{(t^{-1/(N-1)})^{1-N}}{N-1}K \\ &= 2tK/(N-1).\end{aligned}$$

Thus $\|\gamma_t(\Phi') - \Phi'\| \leq 4tK/(N - 1)$, so that $\|\gamma_t(\Phi') - \Phi'\|/t^C$ is bounded for $t \leq 1$. At the same time we have

$$|(\gamma_t(\Phi_q) - \Phi_q)(x, y, p)| = |(e^{iqt} - 1)\Phi_q(x, y, p)| \leq |qt|\|\Phi\|_\infty \leq (t + t^{1-1/(N-1)})\|\Phi\|_\infty$$

for $|q| \leq t^{-1/(N-1)} + 1$, so that for $t \leq 1$ and $|q| \leq t^{-1/(N-1)} + 1$ we have

$$\|\gamma_t(\Phi_q) - \Phi_q\| \leq 2t^{1-1/(N-1)}\|\Phi\|_\infty.$$

Hence

$$\|\gamma_t(\Phi - \Phi') - (\Phi - \Phi')\| \leq (2t^{-1/(N-1)} + 3) \cdot 2t^{1-1/(N-1)}\|\Phi\|_\infty \leq 12t^C\|\Phi\|_\infty.$$

We conclude that $\|\gamma_t(\Phi) - \Phi\|/t^C$ is bounded for $t \leq 1$. But for $t \geq 1$ we have

$$\|\gamma_t(\Phi) - \Phi\| \leq 2\|\Phi\| \leq 2t^C\|\Phi\|,$$

so that $\|\gamma_t(\Phi) - \Phi\|/t^C$ is bounded for all $t > 0$. Thus $\Phi \in {}_{\hbar}^{A,B,C}$. (c). This is proved in the same way as Theorem 15. □

Theorem 3.10.13. *${}_{\hbar}$ and ${}_{\hbar}^{1,1,1/2}$ are identical as sets and have isomorphic norms.*

Proof. ${}_{\hbar}^{1,1,1/2} \subset {}_{\hbar}$ is clear from the definitions. Conversely, it was noted in the proof of Lemma 14 that if $x \in \mathrm{ball}({}_{\hbar})$ then $\|\gamma_t(x) - x\| \leq 4\sqrt{t/c}$. Hence $\|\gamma_t(x) - x\|/\sqrt{t}$ is bounded and we get $x \in {}_{\hbar}^{1,1,1/2}$. So ${}_{\hbar} = {}_{\hbar}^{1,1,1/2}$ as sets.

Isomorphism follows from the estimate

$$\|x\|_L \leq \max(\|x\|, \|\delta_1(x)\| + \|\delta_2(x)\|) \leq \max(\|x\|, 2L(x)) \leq 2\|x\|_{A,B,C}$$

together with the open mapping theorem. □

As a final topic we consider a natural Laplacian and the heat semigroup it generates. The following tool is needed.

Definition 10. *Define* $\tau : N_\hbar \to \mathbb{C}$ *by*

$$\tau(T) = \frac{1}{2\pi}\int_0^1 \int_0^{2\pi} \Phi(x, y, 0)dxdy$$

where Φ *is the function associated to* $a_0(T)$ *in Lemma 5.*

Theorem 3.10.14. *[164, p558] The map* τ *is a faithful normal finite trace. It is invariant for the action* L *given in Definition 7.*

(The fact that τ is normal follows from the fact that it is the composition of two normal maps: integration of $\gamma_t(T)$ and integration over $\mathbf{T}^2$.)

Proposition 3.10.15. *The GNS representation of* $N_\hbar$ *associated to* τ *is unitarily equivalent to the restriction of its original representation on* $H = L^2(\boldsymbol{R} \times \boldsymbol{T} \times \boldsymbol{Z})$ *to* $H' = L^2([0, 2\pi] \times \boldsymbol{T} \times \boldsymbol{Z})$.

Proof. Define a map $\Psi : N_\hbar \to H'$ by $\Psi(T) = T(\xi)$ where $\xi(x, y, p) = 1$ for $(x, y, p) \in [0, 2\pi] \times \mathbf{T} \times \{0\}$ and $\xi(x, y, p) = 0$ elsewhere. Then for any $T \in N_\hbar$ we have $\tau(T) = \langle T\xi, \xi\rangle$, so that Ψ extends to a unitary map from the Hilbert space of the GNS representation onto H'. It intertwines the action of $N_\hbar$ because

$$\Psi(TS) = TS(\xi) = T\Psi(S).$$

Now we define a Laplacian on S^c, and apply a theorem of Sauvageot to establish the existence of a heat semigroup. □

Definition 11. *For* $\Phi \in S^c$ *define* $\Delta\Phi$ *by* $\Delta\Phi = \delta_1(\delta_1(\Phi)) + \delta_2(\delta_2(\Phi))$. *Concretely, we have*

$$\Delta\Phi = \Phi_{xx} - p^2c^2x^2\Phi(x, y, p) - 2ipcx\Phi_y + \Phi_{yy}.$$

The fact that $\Delta\Phi \in S^c$ *follows immediately from the first definition of* $\Delta\Phi$, *or is a routine computation from the second.*

Theorem 3.10.16. *The operators* $e^{-t\Delta}$, $t \geq 0$, *form a weak-operator continuous semigroup of completely positive normal contractions of* $N_\hbar$.

Proof. First we show that $\delta_1 \oplus \delta_2$ is closed when regarded as an operator from H' into $H' \oplus H'$. Identify H' with the functions Φ on $\mathbf{R} \times \mathbf{T} \times \mathbf{Z}$ which satisfy

$$\Phi(x + k, y, p) = e^{ickpy}\Phi(x, y, p)$$

for all $k \in 2\pi\mathbb{Z}$ and whose restriction to $[0, 2\pi] \times \mathbf{T} \times \mathbf{Z}$ is square integrable. Then for all $r, s, t \in \mathbb{R}$ the formula

$$(U_{(r,s,t)}\xi)(x, y, p) = e^{ip(t+cs(x+\hbar p\mu-r))}\xi(x - r, y - s, p)$$

from Definition 7 defines a unitary operator on H'. Thus setting $\alpha'_r(\xi) = U_{(r,0,0)}\xi$ and $\beta'_s(\xi) = U_{(0,s,0)}\xi$ gives us two strongly continuous one-parameter unitary groups on H', whose generators D_1 and D_2 satisfy $D_i(\Phi) = \delta_i(\Phi)$, treating Φ respectively as an element of H' and of $N_\hbar$.

Since D_1 and D_2 are self-adjoint they are closed, hence

$$D = D_1 \oplus D_2 : H' \mapsto H' \oplus H'$$

is closed. Also $D^*(\xi \oplus 0) = D_1(\xi) = \delta_1(\xi)$ and $D^*(0 \oplus \xi) = D_2(\xi) = \delta_2(\xi)$, so

$$D^*D(\Phi) = D^*(\delta_1(\Phi) \oplus \delta_2(\Phi)) = \delta_1(\delta_1(\Phi)) + \delta_2(\delta_2(\Phi)) = \Delta\Phi$$

for all $\Phi \in S^c$. □

Chapter 4

Last Articles

4.1 A Geometric Characteristic Splitting in all Dimensions

by Bernhard Leeb [*] and Peter Scott[†].

Preprint - May 20, 1996

Abstract:
A non RNP Banach space E is constructed such that E^* is separable and RNP is equivalent to PCP on the subsets of E.

The problem of the equivalence of the Radon-Nikodym Property (RNP) and the Krein Milman Property (KMP) remains open for Banach spaces as well as for closed convex sets. A step forward has been made by Schachermayer's Theorem[179]. That result states that the two properties are equivalent on strongly regular sets. Rosenthal, [180], has shown that every non-RNP strongly regular closed convex set contains a non-dentable subset on which the norm and weak topologies coincide. In a previous paper ([181]) we proved that every non RNP closed convex contains a subset with a martigale coordination. Furthermore we established the $P\alpha\ell$-representation for several cases. The remaining open case in the equivalence of RNP and KMP is that of B-spaces or closed convex sets where RNP is equivalent to PCP in their subsets. Typical example for a such structure are the subsets of $L^1(0,1)$. H. Rosenthal raised the question if this could occur when the dual of the space is separable. W. James [182] also posed a similar problem. The aim of the present paper is to give an example of a Banach space E with separable dual failing RNP, and RNP is equivalent to PCP on its subsets. As consequence we get that E does not contain $c_o(\mathbb{N})$ isomorphically and hence it does not embed into a Banach space with an unconditional skipped F.D.D. On the other hand E semiembeds into a Banach space with an unconditional basis. The last property allows us to conclude that every closed convex non-RNP subset of E contains a closed non-dentable set with a $P\alpha\ell$-representation. We recall that a closed set K has a $P\alpha\ell$-representation if there is an affine, onto, one to one continuous map from the atomless probability measures on $[0,1]$ to the set K. In particular RNP is equivalent to KMP on the subsets of E. The space E is realized by applying the Davis-Figiel-Johnson- Pelczynski factorization method to a convex symmetric set W of a Banach space E_u constructed in this paper.

Finally as a consequence of the methods used in the proofs of the example we obtain that

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[*] Partially supported by SFB 256 (Bonn) and IHES

[†] Partially supported by NSF grant DMS 9306240

every separable B-space X such that X^{**}/X is isomorphic to $\ell^1(\Gamma)$ has RNP. We thank H. Rosenthal and T. Odell for some useful discussions related to the problem studied in the present paper. We also thank the Department of Mathematics of Oklahoma State University for its technical support.

We start with some definitions, notations and results necessary for our constructions. A closed convex bounded set K is said to be δ-non dentable, $\delta > 0$, if every slice of K has diameter greater than δ. A closed convex set has RNP if it contains no δ-non dentable set. A closed K subset of a B-space has the P.C.P. if for every subset L of K and for all $\varepsilon > 0$ there exists a relatively weakly open neibhd of L with diameter less than ε. It is well known that RNP implies P.C.P, but the converse fails [183].

In the sequel $\mathcal{D}$ denotes the dyadic tree namely the set of all finite sequences of the for $a = \{0, \varepsilon_1, ..., \varepsilon_n\}$ with $\varepsilon_i = 0$ or 1. For a in $\mathcal{D}$ the length of α is denoted by $|a|$. A natural order is induced on $\mathcal{D}$, that is $a \prec \beta$ if the sequence a is an initial segment of the sequence β. Two elements a, β of $\mathcal{D}$ are called **incomparable** if they are imcomparable in the above defined order. We notice, for later use, that each a in $\mathcal{D}$ determines a unique basic clopen subset V_a in Cantor's group $\{0,1\}^{\mathbb{N}}$ and a, β are imcomparable if $V_a \cap V_\beta = \emptyset$.

A basic ingredient in the definition of the space E is Tsirelson's norm as it is defined in [184]. We recall that the norm of this space satisfies the following implicit fixed point property.

For $x = \sum_{\kappa=1}^{m} \lambda_\kappa t_\kappa$

$$||\sum_{K=1}^{m} \lambda_\kappa t_\kappa||_T = max\{\max_\kappa |\lambda_\kappa|, \frac{1}{2} sup \sum_{j=1}^{n} ||E_j x||_T\}$$

where the "sup" is taken over all choices

$$m < E_1 < E_2 < ... < E_n$$

$E_1, ..., E_n$ is an increasing sequence of intervals in the set of natural numbers and $E_j x$ is the natural projection of x in the space generated by vectors of the basis $\{t_k :_k \in E_j\}$ Tsirelson's space is a reflexive Banach space with an unconditional basis not containing any ℓ^p for $1 < p < \infty$.

4.1.1 The space $\mathbf{E_u}$

The space E_u will be defined to have an unconditional basis indexed by the dyadic tree $\mathcal{D}$ and denoted by $(e_a)_{a\in\mathcal{D}}$. For a sequence of reals $(\lambda_\alpha)_{\alpha\in\mathcal{D}}$ which is eventually zero we define

$$||\sum_{a\in\mathcal{D}} \lambda_\alpha e_\alpha|| = \sup\{||\sum_{i=1}^{\ell} \lambda_{a_i} t_{k_i}||_T : \{\alpha_i\}_{i=1}^{\ell} \text{ are incomparable,}$$
$$|a_i| = \kappa_i,\ \kappa_1 < \kappa_2 < ... < \kappa_\ell\}.$$

It is clear that $(e_a)_{\alpha\in\mathcal{D}}$ is an unconditional basis for the space E_u defined by the above norm. Next we verify certain properties of the space E_u.

Proposition 4.1.1. *The dual of the space E_u is separable.*

Proof. The spare E_u has an unconditional basis hence it is enough to show that ℓ^1 does not embed into E_u [185]. ☐

Suppose, on the contrary, that ℓ^1 embeds into E_u. Then, by standard arguments, we can find $\ell_1 < \ell_2 < ... < \ell_k < ...$ an increasing sequence of natural numbers and $\{x_k\}_{\kappa=1}^{\infty}$ a normalized sequence in E_u equivalent to the usual basis of ℓ^1, and

$$x_\kappa = \sum_{\ell_\kappa < |\alpha| < \ell_{\kappa+1}} \lambda_\alpha e_\alpha$$

The definition of the norm of E_u and elementary properties of Tsirelson's norm show that

$$||\sum_{k=1}^{m} \mu_k x_k|| \leq ||\sum_{k=1}^{m} {}^{\mu}_{\kappa} t_{\ell_{k+1}}||_T$$

so $\{t_{\ell_k}\}_{k=2}^{\infty}$ is equivalent to the basis ℓ^1. This contradicts to the reflexivity of T.

A consequence of the above Proposition is that the basis $(e_\alpha)_{\alpha\in\mathcal{D}}$ is shrinking. Therefore every x^{**} in E_u^{**} has a unique representation as

$$x^{**} = w^* \lim_{n\to\infty} \sum_{|\alpha|\leq n} \lambda_\alpha e_\alpha := w^* - \sum_{\alpha\in\mathcal{D}} \lambda_\alpha e_\alpha$$

and $\lambda_\alpha =< x^{**}, e_\alpha{}^* >$. We define the **support** of x^{**}, denoted by supp x^{**}, to be the set

$$\{\alpha \in \mathcal{D} :< x^{**}, e_\alpha^* >\neq 0\}.$$

Lemma 4.1.2. *Given* x_1^{**} *,...,* x_κ^{**} *in* E_u^{**} *such that there are* $a_1, ..., a_k$ *incomparable elements of* $\mathcal{D}$ *so that supp* x_i^{**} *is contained in of* $W_{\alpha_i} = \{\beta \in \mathcal{D} : \beta \prec a_i$ *or* $a \prec \beta\}$.

Then

$$d(x_1^{**} + .. + x_\kappa^{**}, E_u) \geq \frac{1}{2}\sum_{i=1}^{\kappa} d(x_1^{**}, E_u)$$

Proof. For $n < m$ we define

$$P_{[n,m]}(x^{**}) = \sum_{n\leq|a|\leq m} \lambda_\alpha e_\alpha$$

and

$$P_{[n,\infty]}(x^{**}) = \sum_{n\leq|\alpha|} \lambda_\alpha e_\alpha$$

where $\lambda_\alpha =< x^{**}, e_a^* >$. ☐

Using this notation we have

$$d(x^{**}, E_u) = \lim_{n\to\infty} ||P_{[n,\infty]}(x^{**})||$$

and

$$||P_{[n,\infty]}(x^{**})|| = \lim_{m\to\infty} ||P_{[n,m]}(x^{**})||$$

To establish the result it is enough to show that for $\epsilon > 0$ there exists $n(\varepsilon)$ such that for all $m > n(\varepsilon)$

$$||P_{[m,\infty]}(\sum_{i=1}^{k} x_i^{**})|| \geq \frac{1}{2}\sum_{i=1}^{k} d(x_i^{**}, E_u) - \epsilon.$$

Actually $n(\varepsilon) = max\{\kappa, |a_1|, ...|a_\kappa|\}$. Choose any $m > n(\varepsilon)$. Inductively we define $\{q_i, \ell_i]_{i=1}^{\kappa}$ such that

$$m < q_1 < \ell_1 < .. < q_k < \ell_k$$

and $||P_{[q_i,\varrho_i]}(x_i^{**})|| > d(x_i^{**}, E_u) - \frac{\in}{2^i}$.

For each $1 \leq i \leq k$ there is a set $\{\beta_j^i\ 1 \leq j \leq s(i)\}$ of incomparable elements of $\mathcal{D}$ such that $q_i \leq |\beta_j^i| \leq \ell_i$ and

$$||P_{[q_i,\ell_i]}(x_i^{**})|| = ||\sum_{j=1}^{s(i)} \lambda_{\beta_j^i} t_{|\beta_j^i|}||_T.$$

Notice that $a_i \prec \beta_j^i$ for all $j = 1...s(i)$. Observe that $\cup_{1\leq i\leq k}\ \{\beta_j^i : 1 \leq j \leq s(i)\}$ consists of pairwise imcomparable elements. So

$$||P_{[m,\infty]}(\sum_{i=1}^{\kappa} x_i^{**})|| \geq ||P_{[m,\ell_k]}(\sum_{i=1}^{\kappa} x_i^{**})|| \geq$$

$$||\sum_{i=1}^{\kappa}\sum_{j=1}^{s(i)} \lambda_{\beta_j^i} t_{|\beta_j^i|}||_T \geq \frac{1}{2}\sum_{i=1}^{\kappa} ||\sum_{j=1}^{s(i)} \lambda_{\beta_j^i} t_{|\beta_j^i|}||_T$$

$$= \frac{1}{2}\sum_{i=1}^{\kappa} ||P_{[q_i,\ell_i]}(x_i^{**})|| \geq \frac{1}{2}\sum_{i=1}^{\kappa} d(x_i^{**}, E_u) - \epsilon \quad \square$$

Consider the following closed convex subset of the unit ball of E_u

$$K = \{x \in E_u : x = \sum_{n=0}^{\infty}\sum_{|\alpha|=n} \lambda_\alpha e_\alpha, \lambda_0 = 1, \lambda_a \geq 0, \lambda_\alpha = \lambda_{(\alpha,0)} + \lambda_{(\alpha,1)}\}.$$

It is easily verified that K is the closed convex hull of a $\frac{1}{2}$-tree $(d_\alpha)_{\alpha\in\mathcal{D}}$ where for every a in $\mathcal{D}$ d_α is defined by the conditions $e_\alpha^*\ (d_\alpha) = 1, e_{(\beta,0)}^*(d_\alpha) = e_{(\beta,1)}^*(d_\alpha) = \frac{1}{2}e_\beta^*(d_\alpha)$ and $d_\alpha \in K$.

We set $W = co(K \cup -K)$ and we denote by $\tilde{W}$ its w^* closure in E_u^{**}. Notice that $x^{**} \in \tilde{W}$ if $|e_\alpha^*(x^{**})| \leq 1, e_{(\alpha,0)}^*(x^{**}) + e_{(a,1)}^*(x^{**}) = e_\alpha^*(x^{**})$ for all a in $\mathcal{D}$. Hence we could define a map

$$T : M_1(\{0,1\}^{\mathbb{N}}) \to \tilde{W}$$

with the rule

$$T(\mu) = w^* - \sum_{\alpha\in\mathcal{D}} \mu(V_\alpha) e_a$$

where $V_\alpha = \{\gamma \in \{0,1\}^{\mathbb{N}} : \gamma \restriction |\alpha| = a\}$

Clearly T is one to one and onto. Furthermore

$$||T(\mu)|| \leq sup\{\sum_{i=1}^{k} |\mu(V_{\alpha_i})| : \{\alpha_i\}_{i=1}^{k} \text{ incomparable } \} = ||\mu||.$$

Hence T is extended to a bounded linear operator from $M(\{0,1\}^{\mathbb{N}})$ onto the linear span of $\tilde{W}$ denoted by $<\tilde{W}>$.

4.1.2 The Space E

The space E is the result of the application of Davis-Figiel-Johnson-Pelczynski [186] factorization method to the set W defined above. We give the precise definition and certain properties of the space E.

$$E = \{y \in E_u : |||y||| = (\sum_{n=1}^{\infty} ||y||_n^2)^{\frac{1}{2}} < \infty\}$$

Here $||.||_n$ denotes the Minkowski's gauge of the set $2^n W + \frac{1}{2^n} B_{E_u}$. Let $J : E \to E_u$ be the natural injection. We notice that contains the set W; hence E fails RNP. The operator J satisfies the following properties.

P.1.: $J^{**} : E^{**} \to E_u^{**}$ is one to one and $J^{**}[E^{**}] \cap E_u = J[E]$. As consequence of this property E^* is separable.

P.2.: J is a weak to weak homeomorphism on the bounded subsets of E. This is a consequence of P.1 and it implies that J[187] is closed for all L, closed convex bounded subsets of E. In particular J is a semiembedding.

P.3.: Let L be a closed convex bounded subset of E failing RNP. Then J[187] is non RNP. If not, J [187] is an RNP set, hence for any L-valued operator $S : L^1 \to E$ the operator JoS is representable by a function φ in $L^{\infty}_{J[L]}$. Then the function $\Psi = J^{-1}\varphi$ represents the operator S and K is RNP [183].

P.4.: If L is bounded subset of E and J[L] fails P.C.P. then L fails P.C.P. Indeed, for $\{y_n\}_{n=1}^{\infty}$, y in L such that $J(y_n) \overset{w}{\to} J(y)$ and $||J(y_n) - J(y)|| > \delta > 0$ P.2. ensures that $y_n \overset{w}{\to} y$ and also $||y_n - y|| > \dfrac{\delta}{||J||}$. Hence y is not a point of continuity.

P.5.: $J^{**}[E^{**}] \subseteq \overline{<\tilde{W}>}$ For this, notice that $B_{E^{**}} \subset 2^n \tilde{W} + \frac{1}{2^n} B_{E_u^{**}}$ hence

$$J^{**}[B_{E^{**}}] \subseteq \cap_n (2^n \tilde{W} + \frac{1}{2^n} B_{E_u^{**}}) \subseteq \overline{<\tilde{W}>}.$$

We proceed to the proof of the main property of the space E.

Proposition 4.1.3. *Let K be a closed, convex, bounded, non RNP subset of E. Then K fails P.C.P.*

Proof. Property 3, mentioned before, ensures that J is non RNP closed subset of E_u. Hence for some $\delta > 0$ there exists a convex closed L subset of J which is δ-nondentable. Our goal is to show that every weak neighd in L has diameter greter than $\frac{\delta}{256}$. By a result due to Bourgain [188] it is enough to show that for every $S_1, S_2, ..., S_n$ slices of $\tilde{L}$ there exists x_i^{**} in S_i $i = 1, 2, ..., n$ such that for all $(\lambda_i)_{i=1}^n \in \mathbb{R}_+^n$ $\quad \sum_{i=1}^n \lambda_i = 1$

$$d(\sum_{i=1}^n \lambda_i x_i^{**}, E_u) > \frac{\delta}{256}$$

$\square$

Given $S_1, S_2, ..., S_n$ slices of $\tilde{L}$. Using Lemma 2.7 from[180] we choose $(x^{**}_{\xi,i})_{\xi>\omega_1}$ an uncountable subset of S_i such that

$$d(x^{**}_{\xi,i} - x^{**}_{\zeta_i i}, E_u) > \frac{3\delta}{8} \text{ for } \xi \neq \zeta.$$

Recall that $\tilde{L}$ is a subset of $J^{**}[E^{**}] \subset \overline{< \tilde{W} >}$ and that $T[M\{0,1\}^{\mathbb{N}}]$ is norm dense into $\overline{< \tilde{W} >}$. Hence there are $(\mu_{\xi,i})_{\xi<\omega_1, i\leq n}$ such that

$$||T\mu_{\xi,i} - x^{**}_{\xi,i}|| < \frac{\delta}{256}$$

Also, it is known that $M(\{0,1\}^{\mathbb{N}}) = (\sum_{\gamma<2^\omega} \oplus L^1(\lambda\gamma))_1$

where $\{\lambda_\gamma\}_{\gamma<2^\omega}$ are pairwise singular probability measures on $\{0,1\},^{\mathbb{N}}$ and $L^1(\lambda_\gamma) = L^1[0,1]$ or $L^1(\lambda_\gamma) = \mathbb{R}$.

Therefore

$$\mu_{\xi,i} = \sum_{\gamma<2^\omega} \frac{d\mu_{\xi,i}}{d\lambda_\gamma}$$

where the sum is taken in ℓ_1-norm.

Choose $F_{\xi,i}$ finite subset of 2^ω so that the measure $\mu'_{\xi,i} = \sum_{\gamma\in F_{\xi,i}} \frac{d\mu_{\xi,i}}{d\lambda_\gamma}$ satisfies

$$||T\mu'_{\xi,i} - x^{**}_{\xi,i}|| < \frac{\delta}{256} \qquad (1)$$

In particular for $\xi \neq \zeta$ we get

$$d(T\mu'_{\xi,i} - T\mu'_{\zeta,i}, E_u) > \frac{\delta}{4} \qquad (2)$$

Apply Erdös-Rado's Lemma to the family $\{F_\xi = \cup_{i=1}^n F_{\xi,i}, \xi < \omega_1\}$ and find A uncountable, F finite such that for $\xi \neq \zeta$ in A

$$F_\xi \cap F_\zeta = F.$$

We set $\lambda_F = \sum_{\gamma\in F} \lambda_\gamma$ and for ξ in A

$$\nu_{\xi,i} = \mu'_{\xi,i} - \frac{d\mu'_{\xi,i}}{d\lambda_F}$$

Claim: For all $i = 1, ..., n$ the set $B_i = \{\xi \in A : d(T_{\nu_{\xi,i}}, E_u) \leq \frac{\delta}{16}\}$ is at most countable.

Proof of the Claim. Suppose that for some i the set B_i is uncountable. Then, since $L^1(\lambda_F)$ is separable, there are $\xi \neq \zeta$ in B_i such that

$$||\frac{d\mu'_{\xi,i}}{d\lambda_F} - \frac{d\mu'_{j,i}}{d\lambda_F}|| < \frac{\delta}{16}$$

But then

$$d(T\mu'_{\xi,i} - T\mu'_{\zeta,i}, E_u) < \frac{\delta}{4}$$

which contradicts inequality (2) and this completes the proof of the claim.

Choose $\xi_1 < \xi\ _2 < ... < \xi_n$ in A such that

$$d(T_{\nu_{\xi_i,i}}, E_u) > \frac{\delta}{16} \tag{3}$$

In the rest of the proof we will denote (ξ_i, i) by ξ_i.

Notice that the measures $\nu_{\xi_1}, ...\nu_{\xi_n}, \lambda_F$ are pairwise singular. Choose W_1,..., W_n pairwise disjoint clopen subsets of $\{0,1\}^{\mathbb{N}}$ pairwise disjoint such that for $i = 1, ..., m$

$$||\nu_{\xi_i}|W_i^c|| < \frac{\delta}{128} \text{ and } ||\frac{d\mu_{\xi_i}}{d_{\lambda_F}}| \overset{n}{\underset{j=1}{\cup}} W_j|| < \frac{\delta}{128} \tag{4}$$

We are ready to prove the desired property. Indeed, for $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ we have

$$d(\sum_{i=1}^{n} \lambda_i T\mu'_{\xi_i}, E_u) \geq d(\sum_{i=1}^{n} \lambda_i T\mu'_{\xi_i} \restriction \overset{n}{\underset{j=1}{\cup}} W_j, E_u) \geq$$

$$d(\sum_{i=1}^{n} \lambda_i (T\nu_{\xi_i} \restriction W_i), E_u) - \sum_{i=1}^{n} \lambda_i ||T\nu_{\xi_i} \restriction \underset{j \neq i}{\cup} W_j|| -$$

$$\sum_{i=1}^{n} \lambda_i ||\frac{d\mu_{\xi_i}}{d_{\lambda_F}} \restriction \overset{n}{\underset{j=1}{\cup}} W_j||$$

From Lemma 1.2 we get

$$d(\sum_{i=1}^{n} \lambda_i (T\nu_{\xi_i} \restriction W_i), Eu) \geq \frac{1}{2} \sum_{i=1}^{n} \lambda_i d(T\nu_{\xi_i} \restriction W_i, E_u)$$

and from (3) and (4) we get

$$d(\sum_{i=1}^{n} \lambda_i T\mu'_{\xi_i}, E_u) > \frac{1}{2}\frac{3\delta}{4} - \frac{\delta}{64} = \frac{\delta}{128}$$

Finally from (1) we have

$$d(\sum_{i=1}^{n} \lambda_i x^{**}_{\xi_i}, E_u) > \frac{\delta}{256}$$

So L fails P.C.P., and P.4 ensures that J^{-1}[187] also fails this property. ☐

Remark 39. *The space E does not contain a subspace isomorphic to* $c_o(\mathbb{N})$*. This is because* $c_o(\mathbb{N})$ *contains a non RNP closed convex subset on which norm and weak topologies coincide. Therefore E does not embed into a space with an unconditional skipped block finite dimensional decomposition. The last follows from the fact that E fails P.C.P. and it does not contain* $c_o(\mathbb{N})$*. Finally E semiembeds into* E_u *a space with an unconditional basis.*

Proposition 4.1.4. *The properties RNP and KMP are equivalent on the subsets of E. Furthermore if K is closed convex non RNP subset of E then it contains a subset L with a $P\alpha\ell$-representation.*

Proof. As we mentioned before if K is closed convex bounded non RNP then J carriers the same properties and it is contained into E_u which has an unconditional basis. Therefore, there exists an L closed convex subset of J with a $P\alpha\ell$-representation[181]. Then J^{-1}[187] has the same property. ☐

We conclude with the following result.

Theorem 4.1.5. *Suppose that X is a separable Banach space such that X^{**}/X is isomorphic to $\ell^1(\Gamma)$. Then X has RNP.*

Proof. Assume that X contains a δ-non dentable subset K. Then the techniques developed in the proof of Proposition 1.3 shows that K is non strongly regular. Actually every $\sum_{i=1}^{u} \lambda_i S_i$ convex combination of slices will have diameter greater than $\frac{\delta}{256}$. Hence by a result due to Bourgain [188], ℓ^1 embeds into X^*, and by Pelczynski's Theorem [189] M[0,1] embeds into X^{**}. But then there exists a sequence $(x_n^{**})_{n\in\mathbb{N}}$ weakly convergent to zero and $d(x_n^{**}, X) > \delta$. This contradicts the Schur property of $\ell^1(\Gamma)$. ☐

Remark 40. *Odell in [190] has constructed a separable B-space X with $X^{**}/X \cong \ell^1(2^\omega)$. From a theorem by Lindenstrauss [187] follows that every separable B space X and its dual X^* are of the form Z^{**}/Z for some separable Banach space Z.*

4.2 Level Sets and the Uniqueness of Measures

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Abstract:
A result of Nymann is extended to show that a positive σ-finite measure with range an interval is determined by its level sets. An example is given of two finite positive measures with range the same finite union of intervals but with the property that one is determined by its level sets and the other is not.

4.2.1 Introduction.

The purpose of this note is to extend some results of Leth, Malitz , and Nymann [191–193] and answer some questions raised in these papers. Those papers deal with a uniqueness property exhibited by positive measures whose range is an interval and used techniques from real analysis. The main idea in this paper to change from a purely real analysis approach to the problems to a functional analytic one. This approach reveals and clarifies the issues involved.

The basic problem is the following. Suppose that μ is a σ-finite signed measure on some measurable space $(\Omega, \mathcal{B})$. We will say that a signed measure ν on $(\Omega, \mathcal{B})$ satisfies $(\mathcal{L})$ if for every A and $B \in \mathcal{B}$ such that $\mu(A) = \mu(B) \neq \pm\infty$, $\nu(A) = \nu(B) \neq \pm\infty$.

Remark 41 (Question 1)**.** *Under what assumptions on μ is it true that if ν satisfies $(\mathcal{L})$, $\nu = \alpha\mu$ for some $\alpha \in \mathbb{R}$?*

We will also consider the stronger condition $(\mathcal{O})$ which requires that condition $(\mathcal{L})$ hold and for every A and $B \in \mathcal{B}$ such that $\mu(A) \leq \mu(B)$, $\nu(A) \leq \nu(B)$.

Remark 42 (Question 2)**.** [‡] *Under what assumptions on μ is it true that if ν satisfies $(\mathcal{O})$, $\nu = \alpha\mu$ for some $\alpha \in \mathbb{R}$?*

As a convenience in stating results we will say that μ is *uniquely determined by* (L), respectively, (O), if μ satisfies some property P and for any ν satisfying (L), respectively, (O), ν is a constant times μ.

The condition (L) was used in [193] where it was shown that for finite positive measures with range equal to an interval that μ was uniquely determined by (L). The symmetric form of (O) was considered by Leth for the case of positive purely atomic measures such that for each $\varepsilon > 0$, there are only finitely many atoms of mass greater than ε. He called measures satisfying the symmetric form of (O) sympathetic [§]. This condition arose from some problems in qualitative measure theory. (See the references in [191] for more on this.) Malitz [192] showed that for this same class of atomic measures the symmetric form of (L) implies the symmetric form of (O).

It is easy to give examples where even under the symmetric form of (O) μ is not uniquely determined. As pointed about by Leth [191], if $0 < r < \frac{1}{2}$ and μ is purely atomic with atoms

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[‡] mention by $\mathcal{L}, \mathcal{O}, \mathcal{B} = L, O, B$

[§] all symboles we mention it in remark 2 and bellow.

$\{A_n : n = 0, 1, ...\}$ such that $\mu(A_n) = r^n$, there are no sets $A \neq B$ such that $\mu(A) = \mu(B) < \infty$. In order to generalize this observation, Leth makes the following definition. An atom A is a *bully* if

$$\sup\{\sum \mu(B_j) : (B_j) \subset B, B_j \cap B_k = \emptyset, j \neq k \text{ and } \mu(B_j) < \mu(A) \text{ for all } j\} < \mu(A).$$

In the example just given every atom is a bully. Clearly if μ is purely atomic and every atom is a bully, the condition (L) in vacuous and any purely atomic measure ν for which every atom is a bully will satisfy (O). On the other hand, Leth gives examples to show that bullies can be present but under (L) μ is uniquely determined. Leth (as improved by Malitz) also showed that if the space consists of a sequence of atoms with measures converging to 0 and there are no bullies then μ is uniquely determined by the symmetric form of (L). It is not hard to see that the no bullies condition is equivalent for positive measures to the condition that the range of the measure is an interval and therefore Nymann's result can be viewed as an extension of Leth's in the case of finite measures. Thus the question remains as to what conditions on μ imply that (L) or (O) uniquely determines μ.

In this paper we will complete the work of [193] by showing that a positive σ-finite measure with range an interval is uniquely determined by (L). We will also prove some results on signed measures. Leth asked if it is possible to construct a measure so that if the atoms are listed in decreasing order every second atom is a bully, but the measure is uniquely determined. We will show that this is impossible. We also will give several examples which show that the range of measure is not a good indicator of whether the measure is uniquely determined. In particular we present an example of two positive measures μ and ν such that the range of μ is a finite union of intervals, the range of μ equals the range of ν, μ is uniquely determined by (L) but ν is not uniquely determined by (O).

We will use standard notation and terminology from real analysis and elementary functional analysis as may be found in[194, 195]. By a σ-finite signed measure μ we mean an extended real valued measure on a measurable space (Ω, B) taking on only one of the two infinite values $\pm\infty$ such that $|\mu|$ is σ-finite. If μ is a σ-finite signed measure we will denote the atoms of μ by $\{A_i\}$ and the complement of the union of the atoms by C. If for every $\varepsilon > 0$ there only finitely many atoms A_i with $|\mu|(A_i) > \varepsilon$, we will assume that the atoms are ordered so that $|\mu|(A_i) \geq |\mu|(A_{i+1})$ for all i. L_p will always refer to $L_p(\Omega, B, |\mu|), 1 \leq p \leq \infty$. If R is a subset of a vector space X then sp R will denote the span of R.

4.2.2 A Functional Analytic Approach.

Before we reformulate the problem in functional analytic terms, we will present two simple lemmas which will allow us to assume that the measure μ is positive in most of our considerations. We would like to thank the referee who suggested these to us and made several other suggestions to improve the exposition.

Theorem 4.2.1. *Suppose that μ is a σ-finite signed measure on (Ω, B) and Ω^+ and Ω^- are the positive and negative sets of the Hahn decomposition of μ, i.e., $\mu_{|\Omega^+} \geq 0$, $\mu_{|\Omega^-} \leq 0$ and $\Omega^+ \cup \Omega^- = \Omega$. Suppose that ν is another σ-finite signed measure on (Ω, B) and let*

$$\nu'(A) = \nu(A \cap \Omega^+) - \nu(A \cap \Omega^-)$$

for all $A \in B$. Then ν' is a σ-finite signed measure on (Ω, B) such that ν satisfies (L), repsectively, (O), with respect to μ if and only if ν' satisfies (L), respectively, (O), with respect to $|\mu|$.

Proof. Suppose that (O) is satisfied for μ and ν and that $A, B \in B$ such that

$$|\mu|(A) = \mu(A \cap \Omega^+) - \mu(A \cap \Omega^-) \leq \mu(B \cap \Omega^+) - \mu(B \cap \Omega^-) = |\mu|(B).$$

Then

$$\mu(A \cap \Omega^+) + \mu(B \cap \Omega^-) \leq \mu(B \cap \Omega^+) + \mu(A \cap \Omega^-).$$

By (O),

$$\nu(A \cap \Omega^+) + \nu(B \cap \Omega^-) \leq \nu(B \cap \Omega^+) + \nu(A \cap \Omega^-)$$

and hence

$$\nu'(A) = \nu(A \cap \Omega^+) - \nu(A \cap \Omega^-) \leq \nu(B \cap \Omega^+) - \nu(B \cap \Omega^-) = \nu'(B).$$

The proofs of the converse and the case of (L) are similar. □

Thus μ is uniquely determined by (L), respectively, (O), if and only if $|\mu|$ is uniquely determined by (L), respectively, (O),

Most of the conditons we will impose on μ are on the range of μ and the next lemma shows that the conditions are easily transferred to the range of $|\mu|$.

Lemma 4.2.2. *If μ is a σ-finite signed measure on (Ω, B) and Ω^+ and Ω^- are the positive and negative sets of the Hahn decomposition of μ, then*

$$2\textit{range}\ |\mu| = \textit{range}\ \mu - \mu(\Omega^-), \qquad \textit{if}\ \mu(\Omega^-) > -\infty,$$

and

$$\textit{range}\ |\mu| = -\textit{range}\ \mu + \mu(\Omega^+), \qquad \textit{if}\ \mu(\Omega^+) < +\infty.$$

Proof. Assume that $\mu(\Omega^-) > -\infty$. If $A \in B$,

$$\begin{aligned} \mu(A) = \mu(A \cap \Omega^+) + \mu(A \cap \Omega^-) &= \mu(A \cap \Omega^+) + \mu(\Omega^-) - \mu(\Omega^- \setminus A) \\ &= |\mu|((A \cap \Omega^+) \cup (\Omega^- \setminus A)) + \mu(\Omega^-) \end{aligned}$$

and

$$\begin{aligned} |\mu|(A) = \mu(A \cap \Omega^+) - \mu(A \cap \Omega^-) &= \mu(A \cap \Omega^+) - \mu(\Omega^-) + \mu(\Omega^- \setminus A) \\ &= \mu((A \cap \Omega^+) \cup (\Omega^- \setminus A)) - \mu(\Omega^-). \end{aligned}$$

Thus range $|\mu| =$ range $\mu - \mu(\Omega^-)$. The other case is similar. □

We will now formulate the problem in functional analytic terms. Let μ be a positive σ-finite measure and consider the following linear subspace X of L_0, the space of equivalence classes (a.e. μ) of measurable functions on (Ω, B).

$$X = \mathrm{sp}\{1_A - 1_B : \mu(A) = \mu(B) \neq +\infty, A, B \in B\}.$$

From a functional analytic viewpoint condition (L) describes a subset of the kernel of μ as a linear functional on some subspace of L_0. Two possibilities for the subspace come immediately to mind, L_1 and L_∞. If ν is a σ-finite signed measure satisfying (L), then taking A to be the empty set we see that $\nu \ll \mu$ and that if $g = d\nu/d\mu$ then $\int_\Omega fgd\mu = 0$ for all of $f \in X$. If ν has finite variation then $g \in L_1$ and the uniqueness question for condition (L) is equivalent to determining if $\overline{X}^{w^*} = L_\infty^0$, the set of functions h in L_∞ with $\int h d\mu = 0$. If $d\nu/d\mu \in L_\infty$ then uniqueness under (L) is equivalent to $\overline{X} = L_1^0$. (Here the overbar denotes norm closure.)

If μ is σ-finite but not finite then we need to consider the space of functions with finite L_1 norm on each set of finite measure with the (locally convex) topology of L_1 convergence on sets of finite measure which we denote by FL_1. The dual of FL_1 is $FL_\infty = \{f \in L_\infty : \mu(\{f \neq 0\}) < \infty\}$. Note that $X \subset FL_\infty$ and (L) determines μ uniquely if and only if the $\sigma(FL_\infty, FL_1)$ closure of X in FL_∞ is FL_∞^0. Let X_1 denote the norm closure of X in L_1 and let X_∞ denote the $\sigma(FL_\infty, FL_1)$ closure of X in FL_∞. If μ is finite $X_\infty = \overline{X}^{w^*}$, i.e., the $\sigma(L_\infty, L_1)$ closure.

To see the power of this viewpoint consider the following question posed in [191]. Is it possible to construct a purely atomic measure with atoms (A_i) such that $\mu(A_i) \downarrow 0$ and every second atom is a bully but (O) does uniquely determine μ? Actually we can show that neither A_1 and A_2 can be a bully and have μ uniquely determined. Indeed, A_1 cannot be a bully since we need only choose $\nu(A_1) > \mu(A_1)$ and $\nu(A_i) = \mu(A_i)$ for $i > 1$. If A_1 is not a bully but A_2 is, define a map from L_∞^0 into $\mathbb{R}^2$ by $Tf = (\int_{A_1} f d\mu, \int_{A_2} f d\mu)$. Then the range of T is $\mathbb{R}^2$ but if $f \in X$ then $f = c(1_{A_1} - 1_{A_2}) + g$ where supp $g \subset \cup_{i>2} A_i$. Thus the range of $T_{|X_\infty}$ is a subspace properly contained in $\mathbb{R}^2$. This immediately implies that (L) does not uniquely determine μ. It will follow from Corollary 1.7 c) below that (O) cannot determine μ uniquely, but we can see this directly as follows. Fix b, $0 < b < 1$ and define $\nu(A_i) = b\mu(A_i)$ for $i > 2, \nu(A_1) = \mu(A_1)$, and $\nu(A_2) = (1-b)\mu(A_1) + b\mu(A_2)$. A simple computation shows that ν satisfies (O). The point of the using functional analysis is that it often reduces questions about (L) to counting dimensions.

Next we will explore the role of the non-atomic part of μ.

Lemma 4.2.3. *If μ is a finite purely non-atomic positive measure, then $X_1 = L_1^0$ and $X_\infty = L_\infty^0$.*

Proof. First note that because μ is finite $L_\infty \subset L_1$ and thus by the Hahn-Banach theorem if $X_\infty = L_\infty^0$, $X_1 = L_1^0$. Thus we need only consider the case of X_∞. Observe that L_∞^0 is a w^* closed subspace of L_∞ and thus the unit ball of L_∞^0 is w^* compact. It is easy to see that an extreme point of the unit ball of L_∞^0 is of the form $1_A - 1_B$ where $\mu(A) = \mu(B)$, $A \cup B = \Omega$, and $A \cap B = \emptyset$. Thus all of these extreme points are in X_∞ and it follows from the Krein-Milman theorem that X_∞ is w^* dense in L_∞^0. □

The argument above used the extreme points which happened to be in X. If the measure space contains atoms then the extreme points of the ball of L_∞^0 are of the form $1_D - 1_E + \lambda 1_A$ where D, E and A are disjoint, $D \cup E \cup A = \Omega, A$ is an atom, $|\lambda| \leq 1$, and $\lambda\mu(A) + \mu(D) = \mu(E)$. Whether or not such things are in X_∞ is a matter of the combinatorics of the measures of the atoms, however, we can still say something about the non-atomic part.

Proposition 4.2.4. *If μ is a σ-finite positive measure on (Ω, B), $\mu = \mu_c + \mu_a$, where μ_c has no atoms and μ_a is purely atomic, and ν is a σ-finite signed measure which satisfies (L) then $\frac{d\nu}{d\mu_c}$ is constant.*

Proof. The hypothesis implies that $\nu \ll \mu$ and thus if $A \in B$ such that $\mu(A)$ and $\nu(A)$ are finite, $g 1_A \in L_1$, where $g = \frac{d\nu}{d\mu}$. Because both μ and ν are σ-finite we can find a sequence of sets $D_n \in B$ such that $\mu(D_n)$ is finite and non-zero, $\cup_n D_n$ contains the complement of the union of the atoms, and $g_{|D_n} \in L_\infty(\mu_{|D_n})$ for all n. Now by Lemma 1.3, $X \cap L_1^0(\mu_{c|D_1 \cup D_j})$ is dense in $L_1^0(\mu_{c|D_1 \cup D_j})$ for each j. Thus g is constant on $D_1 \cup D_j$ for all j and therefore g is constant on $\cup D_n$, establishing the result. □

Proposition 1.4 shows that the difficulty really is in the atomic part of μ. Next we show that uniqueness can be established provided it holds for a rich enough family of restrictions of μ.

Proposition 4.2.5. *Suppose that μ is a σ-finite measure and for every pair of disjoint sets D and E of finite measure there is a measurable set F such that $X_\infty \cap FL_\infty(\mu_{|F}) = FL_\infty^0(\mu_{|F})$, respectively, $X_1 \cap FL_1(\mu_{|F}) = FL_1^0(\mu_{|F})$, and there exist D' and E' contained in F such that $\mu(D) = \mu(D')$ and $\mu(E) = \mu(E')$. Then if $g \in FL_1$, respectively, $g \in L_\infty(\mu)$, and $g|_X = 0$, then g is constant.*

Proof. Because $\mathrm{sp}\{\mu(D)1_E - \mu(E)1_D : D, E \text{ disjoint and finite measure}\}$ is dense in FL_∞^0 and in FL_1^0, in either case it is sufficient to show that $\mu(E)\int_D g d\mu = \mu(D)\int_E g d\mu$ for all disjoint sets and D and E of finite measure. So given D and E let F, D', and E' satisfy the hypothesis of the proposition. Because $\mu(D')1_{E'} - \mu(E')1_{D'}$, $1_{D'} - 1_D$, and $1_{E'} - 1_E$ are in X, $\mu(E')\int_{D'} g d\mu = \mu(D')\int_{E'} g d\mu$, $\int_{D'} g d\mu = \int_D g d\mu$, and $\int_{E'} g d\mu = \int_E g d\mu$. Hence

$$\mu(E)\int_D g d\mu = \mu(E')\int_{D'} g d\mu = \mu(D')\int_{E'} g d\mu = \mu(D)\int_E g d\mu. \quad \square$$

□

In the papers [191, 192] the following auxiliary function plays an important role. Let $W =$ range of μ and suppose that ν satisfies (L). For each $w \in W$ define $f(w) = \nu(A)$ if $\mu(A) = w$. The absolute continuity of ν with respect to μ implies that $f(0) = 0$. The lemma below summarizes the important properties of f. Versions of a)-c) are in [191, 192]. Because the arguments are short we include them for the sake of completeness.

Lemma 4.2.6. *Suppose that μ is a σ-finite positive measure and ν is a σ-finite signed measure on (Ω, B). If ν satisfies (L), then with f as above*

1. *"a)"f is well defined and finite for all w finite in W.*
2. *"b)"(O) is equivalent to the assertion that f is non-decreasing.*
3. *"c)" If there exists sets $A \in B$ of arbitrarily small but positive measure, and $f'(w)$ exists at some $w \in W$, then $\lim_{\mu(A)\to 0} \frac{\nu(A)}{\mu(A)}$ exists and equals $f'(w)$. (f' is defined only at limit points of W.) In particular, if (O) holds and W has positive measure then $\lim_{\mu(A)\to 0} \frac{\nu(A)}{\mu(A)}$ exists.*

If range μ is a finite interval, we have in addition

1. *"d)"f is continuous.*
2. *"e)"If $\nu \geq 0$, then f is nondecreasing. Consequently, (O) is satisfied and $f'(w)$ exists and is constant a.e. μ.*

Proof. a) and b) are obvious. Let w be a point in the range of μ where the derivative exists and suppose that $\mu(B) = w$. If $A_n \subset B, \mu(A_n) \neq 0$, and $\mu(A_n) \to 0$ then

$$f'(w) = \lim \frac{\nu(B \setminus A_n) - \nu(B)}{\mu(B \setminus A_n) - \mu(B)} = \lim \frac{-\nu(A_n)}{-\mu(A_n)}.$$

If $A_n \cap B = \emptyset, \mu(A_n) \neq 0$ and $\mu(A_n) \to 0$, then

$$f'(w) = \lim \frac{\nu(B \cup A_n) - \nu(B)}{\mu(B \cup A_n) - \mu(B)} = \lim \frac{\nu(A_n)}{\mu(A_n)}.$$

Finally if (A_n) is any sequence of sets with $\mu(A_n) \to 0$,

$$\begin{aligned}\nu(A_n) - f'(w)\mu(A_n) &= \nu(A_n \cap B) - f'(w)\mu(A_n \cap B) + \nu(A_n \cap B^c) - f'(w)\mu(A_n B^c) \\ &= o(\mu(A_n \cap B)) + o(\mu(A_n \cap B^c)) = o(\mu(A_n)).\end{aligned}$$

If W has positive measure there must be a sequence of sets (A_n) with $\mu(A_n) \to 0$. Indeed, if not, μ would be purely atomic with the measures of the atoms bounded away from 0 and thus the range of μ would be a countable set. If (O) is satisfied then f' exists at almost every point of W.

For d) we will show that continuity at $\mu(B) = x$ follows from the absolute continuity of ν with respect to μ. First if $\mu_c \neq 0$, by Proposition 1.4 there is a constant $\kappa > 0$ such that $\nu = \kappa\mu_c$ or $-\kappa\mu_c$ on $\Omega \backslash \cup A_i$, where (A_i) is the sequence of atoms of μ. Given $\varepsilon > 0$ find δ, $\varepsilon/(3\kappa) > \delta > 0$ such that $|\nu|(A) < \varepsilon/6$ if $\mu(A) < \delta$. Choose n such that $\sum_{i \geq n} \mu(A_i) < \delta/2$. Let $\{r_j\}$ be the range of $\mu|_{\cup_{i<n} A_i}$ and let K be the range of $\mu|_{C \cup \cup_{i \geq n} A_i}$. Because K is compact there is a number $\delta' > 0$ such that if $x \notin r_i + K, (x - \delta', x + \delta') \cap r_i + K = \emptyset$. Let $\rho = \min\{\delta, \varepsilon/(3\kappa), \delta'\}$. If $y \in W$ and $|x - y| < \rho$, then for some i, x and y are in $r_i + K$. Therefore

$$|f(x) - f(y)| = |f(r_i) + \nu(\cup_{i \in I} A_i) + \nu(D) - (f(r_i) + \nu(\cup_{i \in J} A_j) + \nu(E))|$$

where $I \cup J \subset \{i \geq n\}$, $D \cup E \subset C$, $x = r_i + \mu(\cup_{i \in I} A_i \cup D)$, and $y = r_i + \mu(\cup_{i \in J} A_i \cup E)$. Because $\mu(\cup_{i \in I \cup J} A_i) < \delta/2$ and $|\nu(D) - \nu(E)| = \kappa|\mu(D) - \mu(E)|$,

$$|f(x) - f(y)| < \varepsilon/3 + \kappa|\mu(D) - \mu(E)| \leq \varepsilon/3 + \kappa[|x - y| + \delta] < \varepsilon.$$

To see that f is nondecreasing when μ, $\nu \geq 0$, suppose that this is false. Then f must have a local minimum at some point x, $0 < x < \mu(\Omega)$, and we may assume that $f(y) > f(x)$ for all y in some interval $(x - \delta, x)$, $\delta > 0$. Because the range of a finite measure is compact, either $x = \mu(\cup_{i \in I} A_i) + \mu(D)$ with $D \subset C$ and $\mu(D) > 0$, or $x = \sup\{\sum_{i \in I} \mu(A_i) : \sum_{i \in I} \mu(A_i) < x\}$ for some infinite set $I \subset \mathbb{N}$. In either case $f(x) = f(y_n) + \nu(B_n)$ for some sequence (y_n) increasing to x. Since $\nu(B_n) \geq 0$, $f(x)$ cannot be a strict local (left) minimum. □

The results in [191, 193] were obtained by showing that in the case in which W is an interval f' is constant and there is no singular part to ν and thus $f(x) = x$. The function f is unsatisfactory for generalizing the results. We will usually use the Radon-Nikodym derivative of ν with respect to μ in our arguments. The Radon-Nikodym derivative is more indicative of the situation because it retains the information about the underlying sets whereas f mostly reflects properties of the range.

Corollary 4.2.7.

1. *"a)"If μ is a σ-finite measure with range of positive Lebesgue measure such that for every $\varepsilon > 0, \mu(\cup\{A : A$ is an atom and $\mu(A) > \varepsilon\}) < \infty$ and ν is a σ-finite signed measure satisfying (O), then $\frac{d\nu}{d\mu} \in L_\infty(\mu)$.*

2. *"b)"If μ is a positive measure with range a finite interval and ν is a positive measure satisfying (L), then $\frac{d\nu}{d\mu} \in L_\infty(\mu)$.*

3. *"c)" If μ is a positive σ-finite measure , ν satisfies (L) and $\frac{d\nu}{d\mu} \in L_\infty(\mu)$ and is not constant then there is a σ-finite positive measure ν' satisfying (L) and $\frac{d\nu}{d\mu}$ is not constant. If, in addition, the range of μ is a finite interval, then ν' satisfies (O).*

Proof. By Proposition 1.4, $\frac{d\nu}{d\mu}$ is constant on C, the complement of the atoms, and by c) of Lemma 1.6, $\lim_{\mu(A)\to 0}\frac{\nu(A)}{\mu(A)}$ exists. Thus $\frac{\nu(A)}{\mu(A)}$ is bounded for all atoms with $\mu(A)$ sufficiently small. The hypothesis implies that there are only finitely many other atoms, and thus $\frac{d\nu}{d\mu}$ is bounded. Part b) follows from e) of Lemma 1.6 and a). If $\mu \geq 0$ and $\frac{d\nu}{d\mu} \in L_\infty(\mu)$ is not constant and (L) holds then there is a nonzero constant γ such that $\nu' = \mu + \gamma\nu \geq 0$. Clearly ν' also satisfies (L). If the range of μ is an interval, by Lemma 1.6 e) ν' satisfies (O). ☐

Note that Corollary 1.7 c) says that if the range of μ is a finite interval, then the uniqueness question for signed measures with bounded Radon-Nikodym derivative is the same for (L) and (O).

4.2.3 Finite Signed Measures.

Next we will prove some technical results which are useful for establishing uniqueness for the case of signed measures.

Lemma 4.2.8. *Suppose that* μ *is a finite positive measure with atoms* $\{A_n\}$ *arranged so that* $\mu(A_n) \geq \mu(A_{n+1})$ *and let* C *be the complement of the union of the atoms. If* Z *is a subspace of* L_∞ *which contains* $\{1_{A_n} - \mu(A_n)/(\mu(C \cup \cup_{j>n} A_j)1_{C\cup\cup_{j>n}A_j} : n = 1, 2, ...\}$
and $L^0_\infty(C,\mu)$, *then* $\overline{Z}^{w^*} \supset L^0_\infty$.

Proof. It is sufficient to show that

$$\mathfrak{D} = \{\mu(A_1)1_{A_k} - \mu(A_k)1_{A_1} : k = 1, 2, ...\} \cup \{\mu(C)1_{A_1} - \mu(A_1)1_C\} \subset \overline{Z}^{w^*}.$$

Indeed if $g \in L_1$ and $z(g) = 0$ for all $z \in \mathfrak{D}$, then $\mu(A_k)g(A_1) = \mu(A_1)g(A_k)$, for all k and $g(A_1)\mu(C) = \int_C g d\mu$. Because $L^0_\infty(C,\mu) \subset Z$, g is a constant γ on C. It follows that g is γ on Ω.

For each n let $B_n = C \cup \cup_{k>n} A_n$. In this notation we have from the hypothesis that $1_{A_n} - \mu(A_n)/\mu(B_n)1_{B_n} \in Z$ for all n. Now observe that since $B_1 = A_2 \cup B_2$,
$1_{A_1} - \mu(A_1)/\mu(B_1)1_{B_1} + \mu(A_1)/\mu(B_1)(1_{A_2} - \mu(A_2)/\mu(B_2)1_{B_2})$
$= 1_{A_1} - \mu(A_1)/\mu(B_2)1_{B_2} \in Z$.
An easy induction argument shows that $1_{A_1} - \mu(A_1)/\mu(B_k)1_{B_k} \in Z$ for all k. Therefore
$1_{A_1} - \mu(A_1)/\mu(A_n)1_{A_n} = 1_{A_1} - \mu(A_1)/\mu(B_{n-1})1_{B_{n-1}}$
$+ (\mu(A_1)/\mu(B_{n-1}) - \mu(A_1)/\mu(A_n))(1_{A_n} - \mu(A_n)/\mu(B_n)1_{B_n}) \in Z$
for all n. If $\mu(C) = 0$ or there are only finitely many atoms, then our initial observation shows that $\overline{Z}^{w^*} \supset L^0_\infty$. If $\mu(C) \neq 0$ and there are infinitely many atoms, then note that $||1_{A_1} - \mu(A_1)/\mu(B_n)1_{B_n}||_\infty \leq \max\{1, \mu(A_1)/\mu(C)\}$ for all n and that $1_{A_1} - \mu(A_1)/\mu(B_n)1_{B_n} \to 1_{A_1} - \mu(A_1)/\mu(C)1_C$ in the w^* topology. Again the conclusion follows from our initial observation. ☐

Theorem 4.2.9. *Suppose that* μ *is a finite positive measure with atoms* $\{A_i\}$, $\mu(A_i) \geq \mu(A_{i+1})$, *and there is a constant* K *such that for each* i, $m \in \mathbb{N}$ *there is an* $h \in X_1$ *with supp* $h - 1_{A_i} \subset \Omega \setminus \cup_{j\leq m} A_j$ *and* $||h||_1 \leq K\mu(A_i)$. *If* $F \in X_1^\perp \cap L_\infty$ *and* $\lim_{n\to\infty} F(A_n)$ *exists, then* F *is a constant.*

Proof. We will show the hypothesis of Lemma 2.1 is satisfied where $Z = \{z \in L_\infty : \int zF d\mu = 0\}$. Let C be the complement of the union of the atoms. Fix $n \in \mathbb{N}$. Let $g = 1_{A_n} - \mu(A_n)/(\mu(C \cup$

$\cup_{j>n}A_j$)$1_{C\cup\cup_{j>n}A_j}$. For each integer $m>n$ let

$$g_m = h_{mn} - \mu(A_n)/\mu(C\cup\cup_{j>n}A_j)\sum_{s=n+1}^{m} h_{ms}$$

where $h_{ms}\in X_\infty$ satisfies $||h_{ms}||_1 \leq K\mu(A_s)$, $h_{ms|\cup_{j<s}A_j}=0$, $h_{ms|A_s}=1_{A_s}$, and $h_{ms|\cup_{s<j\leq m}A_j}=0$, for $n\leq s\leq m$. Therefore

$$||g_m||_1 \leq K\left[\mu(A_n)+\mu(A_n)/\mu(C\cup\cup_{j>n}A_j)\sum_{s=n+1}^{m}\mu(A_s)\right] \leq 2K\mu(A_n).$$

Suppose that $F\in L_\infty\cap X_1^\perp$ and $\lim F(A_n)=\rho$. If $\mu(C)>0$, by Proposition 1.4, $L_1^0(C,\mu)$ is contained in X_1 and thus F is constant on C. Also note that $F(c)=\rho$ for all $c\in C$, because F is constant on C and if $0<\mu(A_i)\leq\mu(C)$, $1_{C'}-1_{A_i}\in X_1$ where $C'\subset C$ such that $\mu(C')=\mu(A_i)$. Hence

$$\begin{aligned}\left|\int(g-g_m)Fd\mu\right| &= \left|\int_{\cup_{j>m}A_j\cup C}(g-g_m)Fd\mu\right|\\ &\leq ||(F-\rho)|_{(\cup_{j>m}A_j\cup C)}||_\infty||g-g_m||_1 + |\int(g-g_m)\rho d\mu|.\end{aligned}$$

The second term is zero because g and $g_m\in L_\infty^0$ and the first term clearly converges to zero. Since $g_m\in X_1$, $\int g_mFd\mu=0$, for each m. Thus $\int gFd\mu=0$ and therefore $\{F\}^\perp\supset L_\infty^0$ and F is constant as claimed. $\square$

As an application of Theorem 2.2 we will prove Nymann's Theorem. We will need the following lemma whose proof we leave to the reader.

Lemma 4.2.10. *If f and $g\in L_1(\mu)$ for some measure μ and there is a set A such that $f_{|A}=\gamma g_{|A}$ for some $\gamma\in[-1,0]$ and $||f_{|A}||_1\geq||f-f_{|A}||_1$, then $||f+g||_1\leq||g||_1$.*

Theorem 4.2.11. *If μ is a positive measure with range equal to a finite interval, ν is a measure satisfying (L), and ν is positive or ν is a signed measure with $\frac{d\nu}{d\mu}\in L_\infty(\mu)$, then ν is a constant times μ.*

Proof. Assume first that ν is positive. We will verify the hypotheses of Theorem 2.2 for $F=d\nu/d\mu$. By Lemma 1.6, $\lim_{\mu(A)\to0}\nu(A)/\mu(A)=c$ so $\lim_{n\to\infty}F(t_n)=c$, if $t_n\in A_n$ and $\mu(A_n)\to0$. Also F is a constant on C by Proposition 1.4 and therefore $F\in L_\infty(\mu)$. Thus we need only produce the functions h in X_1.

Because the range of μ is an interval for each n there exists a subset M_n of $\{n+1,n+2,...\}$ and $C_n\subset C$ such that $\mu(A_n)=\mu(C_n\cup\cup_{j\in M_n}A_j)$, that is, $h_n=1_{A_n}-1_{C_n\cup\cup_{j\in M_n}A_j}\in X_1$ and $||h_{n|A_n}||=||h_n-h_{n|A_n}||$. (See [[191], Proposition 1].)
CLAIM If k and m are integers, $k<m$, and $g\in X_1$, then there is an $h\in X_1$ such that $h_{|\cup_{n\leq k}A_n}=g_{|\cup_{n\leq k}A_n}$, $h_{|\cup_{k<n\leq m}A_n}=0$, and $||g||_{L_1(\mu)}\geq||h||_{L_1(\mu)}$.

Indeed, suppose that $g|_{A_{k+1}}=a1_{A_{k+1}}$. Let $f=-ah_n$ in Lemma 2.3. Then if $g_1=g+f, g_1|_{\cup_{n\leq k}A_n}=g_{|\cup_{n\leq k}A_n}, g|_{A_{k+1}}=0$, and $||g_1||\leq||g||$. Induction finishes the proof of the claim.

Applying the claim to $g = h_n$ shows that the hypothesis of Theorem 2.2 is satisfied and thus $d\nu/d\mu$ is constant.

For ν signed, we apply the previous argument to ν' from Corollary 1.7 c). It follows that ν' and hence ν is a multiple of μ. $\square$

Remark 43. *This theorem includes Nymann's result. The proof given by Nymann uses the result of Malitz [192] which in turn uses [191]. Thus one virtue of the proof we have given is that it is direct. More important our argument shows that there are two main ingredients to the proof. The ability to solve equations and the "continuity at $\emptyset$" of the Radon-Nikodym derivative. This second ingredient is also the main obstacle to be overcome if the requirements on signed measures are to be relaxed. It would be nice to find a replacement for the argument used in the proof of Corollary 1.7 so that signed measures without bounded Radon-Nikodym derivative could also be treated. In particular we do not know whether a finite signed measure with range an interval is determined uniquely by* (L).

4.2.4 σ-Finite Measures.

In the paper [193] only finite positive measures were considered. Here we are allowing σ-finite measures. At first glance it may appear that the σ-finite case should reduce to the finite case. However we wish to observe that given $t \in$ range μ there may be no set C of finite measure such that range $\mu_{|C}$ is an interval containing t. Consider the following example.

Example 4.2.12. *Let Ω be [0,1] union a sequence of disjoint sets $\{A_n\}$ (also disjoint from [0,1]) and let μ restricted to [0,1] be Lebesque measure and let the A_n's be atoms of measure $1 + 2^{-n}$. Note that the range of μ is $\mathbb{R}^+$ but for every C of finite measure greater than one the range of $\mu_{|C}$ misses some interval* $(1, 1 + 2^{-n})$, *with* $n \in \mathbb{N}$.

Thus it is not possible to reduce the problem to the finite case directly. The next few lemmas will show that the example given above is the prototype for this difficulty.

Lemma 4.2.13. *Let μ be a purely atomic measure with range equal to an interval such that for every $\varepsilon > 0, \mu(\cup\{A_n : A_n$ is an atom and $\mu(A_n) > \varepsilon\}) < \infty$. Then for every $t < \mu(\Omega)$ there is a subset C of Ω such that $\mu(C) \geq t$ and the range of $\mu_{|C}$ is a finite interval.*

Proof. There is nothing to prove if $\mu(\Omega) < \infty$, so assume that $\mu(\Omega) = \infty$. The requirement that the range is equal to a finite interval is equivalent to the assumption that there are no bullies. Let (a_n) be the masses at the atoms arranged in decreasing order. Assume that $a_1 = 1$. We will inductively construct a subsequence of the a_n's without bullies so that the sum is finite.

As a convenience we will say that if E and F are subsets of $\mathbb{N}$ and every element of E is less than every element of F then $F > E$. Let $E_0 = \{1, 2, ...N\}$ where N satisfies $\sum_{i \leq N} a_i > t$. Choose $E_1 > E_0$ such that $\frac{1}{2} \leq \sum_{n \in E_1} a_n \leq \frac{2}{3}$ and $a_n \leq \frac{1}{2}$ for all $n \in E_1$. This is possible because $\sum_{n \geq k} a_n = \infty$ for all k and (a_k) decreases to 0. Next choose $E_2 > E_1$ such that $\frac{1}{3} \geq \sum_{n \in E_2} a_n \geq \frac{1}{4}$ and $a_n \leq \frac{1}{4}$ for all $n \in E_2$. Thus we can continue in this way to construct a sequence of finite subsets of $\mathbb{N}$, (E_j), such that for all $j \geq 1$

1. "a)"if $n \in E_j, a_n \leq 2^{-j}$
2. "b)"$2^{-j+2}/3 \geq \sum_{n \in E_j} a_n \geq 2^{-j}$

3. "c)"$E_{j+1} > E_j$

It is easy to see that $C = \cup\{A_n : n \in E_j, \text{ some } j\}$ is the required set. If $n \in E_j$, then $a_n = \mu(A_n) \leq 2^{-j}$ and $\sum_{k>j} \sum_{n \in E_k} a_n \geq \sum_{k>j} 2^{-k} = 2^{-j}$. Thus A_n is not a bully. ☐

Lemma 4.2.14. *If μ is a positive σ-finite measure with range equal to an interval, then there is a possibly infinite number β and a subset B of Ω such that*

1. *"a)"The range of $\mu_{|B}$ is the interval $[0, \beta]$ and for every $t < \beta$ there is a set $B' \subset B$ with the range of $\mu_{|B'}$ an interval of finite length at least t.*
2. *"b)"either*
   (a) *i) $B = \Omega$*
   (b) *" "or*
   (c) *ii) $\mu_{|\Omega \setminus B}$ is purely atomic, every atom in $\Omega \setminus B$ has measure strictly greater than β and β is a limit point of the range of $\mu_{|\Omega \setminus B}$.*

Proof. The result is trivial if $\mu(\Omega) < \infty$. If $\mu(\Omega) = \infty$ there are two cases to consider.

1. "1)"there is a sequence of atoms (A_i) such that $\lim \mu(A_i) = 0$ and $\sum \mu(A_i) = \infty$
2. "2)"there is no such sequence.

If Case 1 occurs, let $A = \cup A_i$, then range of $\mu_{|A} = \mathbb{R}^+$ and by Lemma 3.1 we can take $B = \Omega$ and $\beta = \infty$.

Now we will treat Case 2. As usual let C be the complement of the atoms and let $\kappa = \mu(C)$, which we assume is finite. (If $\kappa = \infty$, we can again let $B = \Omega$.) Let $\mathbf{B} = \{G : G \subset \Omega \setminus C$, $\mu(G) < \infty$ and range $\mu_{|G}$ omits no intervals of length greater than $\kappa\}$. Note that if $B = \cup_{i \in M} G_i$, $G_i \in \mathbf{B}$, for each $i \in M$, and $\mu(B) < \infty$, then range $\mu_{|B}$ omits no intervals of length greater than κ. Indeed, if $H \subset B$ and $s = \sup\{\mu(D) : D \subset B,\ \mu(D) < \mu(H)\} < \mu(H) - \kappa$, then every non-null $E \subset H$ has measure greater than κ. Let E be an atom of minimal measure contained in H, then there exists $G_i \supset E$ with range $\mu_{|G_i}$ missing no interval of length greater than κ. But for some $t \in [\mu(E) - \kappa, \mu(E))$ there is a subset F of G_i so that $\mu(F) = t$. Clearly $F \cap H = \emptyset$ so $\{\mu((H \setminus E) \cup F) : F \subset G\}$ contains a point in $(s, \mu(H))$. Hence $\mu_{|B}$ omits no intervals of length greater than κ.

It follows that if $B_1 = \cup\mathbf{B} \cup C$, then if $t < \mu(\cup\mathbf{B}) + \kappa$ there is a subset B' of $\cup\mathbf{B}$ such that range $\mu_{|B' \cup C}$ is an interval containing [0,t] and $\mu(B' \cup C\ ) < \infty$. If $\mu(B_1) = \infty$, then let $B = \Omega$. Otherwise let $B = B_1$ and $\beta = \mu(\cup\mathbf{B}) + \kappa$. Also note that $\beta > 0$. Indeed if $\beta = 0, 1)$ Case 1 does not occur and there is some $\varepsilon > 0$ so that $\mu(F) < \infty$ where $F = \cup\{A : A \text{ is an atom and } \mu(A) < \varepsilon\}$. Because the range of μ is an interval, $F \neq \emptyset$. The range of $\mu_{|F}$ cannot be an interval because $\beta = 0$, so there must be a bully in F. However this implies that the range of μ is not an interval.

Clearly the range of $\mu_{|B} = [0, \beta]$. Observe that if A is an atom with $\mu(A) \leq \beta < \infty$ then $A \cup B \in \mathbf{B}$ and hence $A \subset B$. Therefore if $\beta < \infty$ and the range of μ is $[0, \infty]$, every atom $A \subset \Omega \setminus B$ has measure greater than β and there must be an infinite sequence of atoms (A_i) such that $\mu(A_i)$ decreases to β. ☐

We are now in a position to extend the result in [193] to all positive σ-finite measures with range an interval.

Proposition 4.2.15. *Suppose that μ is a positive σ-finite measure with range an interval and that ν is a σ-finite measure satisfying (L). If ν is positive or if there exist $\epsilon > 0$ and $M < \infty$ such that $\frac{|\nu(A)|}{\mu(A)} < M$ if $\mu(A) < \epsilon$, then ν is a multiple of μ.*

Proof. By Theorem 2.4, either condition implies that if $B \in \mathcal{B}$, $\mu(B) < \infty$ and range $\mu_{|B}$ is an interval, then $\frac{d\nu}{d\mu}$ is constant on B. Hence we assume that $\mu(\Omega) = \infty$. From Lemma 3.2 it follows that if $\beta = \infty$ then for any pair of disjoint sets of finite measure there is a subset $B^{'} \subset B$ such $\mu_{|B'}$ has range a finite interval containing $\mu(D)$ and $\mu(E)$. By Proposition 1.5 and the finite case this implies that $d\nu/d\mu$ is constant.

Now assume that $\beta < \infty$. In view of the Proposition 1.5 it is sufficient to show that for $r > s > 0$ there is a set F such that $\overline{X \cap L_\infty(\mu_{|F})}^{w^*} = L^0_\infty(\mu_{|F})$, the range of $\mu_{|F}$ contains $\{s, r\}$ and $\mu(F) < \infty$. If $\beta \geq r$ we can take $F = B$ and proceed as above. If $r > \beta$, we will add some atoms to B to get the required set F. From Lemma 3.2 we know that there is an infinite sequence of atoms (A_i) such that $\mu(A_i)$ decreases to β. It follows that there is a finite set of the A_i's, B_1, B_2, ... B_n, such that $\mu(B_i) > \mu(B_{i+1})$ for all i, $\mu(B_1) + \beta > \mu(B_2) + \mu(B_3) > \mu(B_1)$, and both r and s are in the range of $\mu_{|F}$, where $F = B \cup \cup_{i \leq n} B_i$. We will show that X_∞ is w^* dense in $L^0_\infty(\mu_{|F})$. Because the range of $\mu_{|B}$ is a finite interval, $X_\infty \cap L^0_\infty(\mu_{|B})$ is w^* dense in $L^0_\infty(\mu_{|B})$. Also dim $L^0_\infty(\mu_{|F})/L^0_\infty(\mu_{|B}) = n$, so it is sufficient to find n linearly independent elements of this quotient which arise from X. For each $i < n$ there is a subset C_i of B with positive measure such that $1_{B_i} - 1_{B_n} - 1_{C_i} \in X$, and there is a set $C_n \subset B$ such that $1_{B_1} + 1_{C_n} - 1_{B_2} - 1_{B_3} \in X$. It is easy to see that these are in n linearly independent cosets of $\overline{X \cap L_\infty(\mu_{|F})}^{w^*} / L^0_\infty(\mu_{|B})$. $\square$

Remark 44. *Recently Khamsi and Nymann [196] have shown that if μ is a finite measure with range equal to an interval then any σ-finite measure ν such that if $\mu(A) = \mu(B)$ then $\nu(A) = \nu(B)$, i.e., (L) without the finiteness requirement is satisfied, then (L) holds. It follows that Proposition 3.3 holds with the weakened version of (L) as well.*

Theorem 4.2.16. *Suppose that μ is a σ-finite signed measure with range an interval and that ν is a σ-finite measure satisfying (L). If there exist $\epsilon > 0$ and $M < \infty$ such that $\left|\frac{\nu(A)}{\mu(A)}\right| < M$ if $\mu(A) < \epsilon$ and A is an atom, then ν is a multiple of μ.*

Proof. By Lemmas 1.1 and 1.2, $|\mu|$ and ν' also satisfy the hypothesis. By Proposition 3.3, ν' is a multiple of $|\mu|$ and hence ν is a multiple of μ. $\square$

Corollary 4.2.17. *Suppose that μ and ν are σ-finite signed measures satisfying (O). If the range of μ is an interval, then ν is a multiple of μ.*

Proof. Proceeding as in the previous proof, $|\mu|$ and ν' satisfy (O) and by Lemma 1.6 b), ν' is positive. Therefore by Proposition 3.3, ν' is a multiple of $|\mu|$ and hence ν is a multiple of μ. $\square$

4.2.5 Examples.

In order to construct examples we will use the finite dimensional case to control the combinatorics in the same spirit as the examples constructed by Leth. Let us now consider the finite dimensional case, i.e., suppose that there are finitely many atoms $(A_i)_{i=1}^n$ with $\mu(A_i) = a_i$ and assume that

$\Omega = \cup A_i$. The statement of the problem can be rephrased in terms of dimension and thus the uniqueness question for (L) reduces to linear algebra. To make the statement more succinct let us introduce the following notation. Let $R = \{\nu \in \{-1, 0, 1\}^n : \langle \nu, (a_i) \rangle = 0\}$.

Proposition 4.2.18. *Condition (L) uniquely determines μ if and only if dim sp $R = n - 1$.*

It would be nice if there were some easy characterization of the finite sequences (a_i) so that dim sp $R = n-1$, but as we will see below there is no obvious description. One easy observation is that the a_i's must all belong to the same rational equivalence class. (This same observation also applies in the case where μ is purely atomic σ-finite but the measures of the atoms are bounded away from 0.) Note also that uniqueness for (O) implies uniqueness for (L) in the case of finitely many atoms because the Radon-Nikodym derivative is automatically bounded.

For our first example we will produce a finite dimensional example with range not an arithmetic progression but such that (L) uniquely determines μ. (In the closing remarks in [N] it was claimed that no such example exists.) For convenience we will make our measure integer valued.

Example 4.2.19. *Let μ have nine atoms with measures 1, 2, 5, 6, 7, 8, 9, 10, 11. With the notation above and $a_1 = 1, a_2 = 2, ..., a_9 = 11$, the set R contains the following elements.*

(1, 0, 1,-1, 0, 0, 0, 0, 0)
(1, 0, 0, 1,-1, 0, 0, 0, 0)
(1, 0, 0, 0, 1,-1, 0, 0, 0)
(1, 0, 0, 0, 0, 1,-1, 0, 0)
(1, 0, 0, 0, 0, 0, 1,-1, 0)
(1, 0, 0, 0, 0, 0, 0, 1,-1)
(0, 1, 1, 0,-1, 0, 0, 0, 0)
(0, 0, 1, 1, 0, 0, 0, 0,-1)

It is not hard to see that these are linearly independent and therefore by Proposition 4.1 μ is uniquely determined by (L). Clearly 4 is omitted from the range of μ but 1, 2 and 3 are included, so the range is not an arithmetic progression.

Our next two examples show that properties of the range are not good indicators of whether uniqueness holds.

Example 4.2.20. *We begin with two finite dimensional examples. Let the five atoms of μ be of size 1, 2, 2, 2, 5 and let the four atoms of μ' be of size 1, 2,4, 5. The range of μ is the same as the range μ', namely $\{1, 2, ..., 12\}$. For μ note that the set R contains the four linearly independent elements (1, 1, 1, $0, -1$), (0, 1,-1, 0, 0), (0, 0, 1,-1, 0), and (1,-1,-1,-1, 1). Thus μ is uniquely determined by (L). On the other hand for μ' we have $R = \{\pm$(1, 0, 1,-1), $\pm$(1,-1,-1, 1)$\}$ and thus dim sp $R = 2 < 3$. Thus μ' is not uniquely determined by (L). (Take ν to have atoms of size 1, 2, 6, and 7, for example.)*

Example 4.2.21. *We will modify the measures in Example 2 to get two measures with range the same finite union of intervals, but so that uniqueness holds for the first but not the second. To do this observe that if we omit the first atom from each of the measures in Example 2, the range of the restricted measure in each case is $\{$2, 4, 5, 6, 7, 9, 11 $\}$. Let γ be any probability measure on a measurable space (Ω_1, C) with range [0,1]. Replace the atom A_1 by Ω_1 and enlarge* *the σ-algebra to contain C for each of μ and μ' and define $\mu(A) = \mu'(A) = \gamma(A)$ for $A \in C$. (Extend μ and μ' to the σ-algebra in the obvious way.) Then the range of μ and μ' is now* $[0,12] \setminus ((1,2) \cup (3,4) \cup (8,9) \cup (10,11))$. *Proposition 1.2 implies that if ν is another finite measure on $\Omega_1 \cup \cup_{i>1} A_i$ satisfying (L) for either μ or μ' then ν is a multiple of γ on Ω_1. Thus the uniqueness problem is the same here as in Example 2, i.e., for μ condition (L) implies uniqueness but for μ' it does not.*

These examples indicate that a characterization of the measures for which (L) is sufficient for uniqueness would require a similar result for the finite dimensional case and thus seems to be difficult. Finally we present an example of a finite signed measure with range the interval [-1,1] which is uniquely determined by (L) but for which neither the positive nor negative parts are uniquely determined by (O).

Example 4.2.22. *For each natural number n let A_{2n-1} and A_{2n} be atoms of a signed measure μ with $\mu(A_{2n-1}) = \frac{2}{3^n} = -\mu(A_{2n})$. It is not hard to see that the ranges of μ^+ and μ^- are both the usual Cantor set C in [0,1]. Moreover it is known and elementary to see that $C - C = [-1,1]$. By Lemma 2.1, $|\mu|$ is uniquely determined by (L). Indeed, for each n, $1_{A_{2n-1}} - 1_{\cup_{j>2n} A_j}$ and $1_{A_{2n}} - 1_{\cup_{j>2n} A_j} \in X$. Also $1_{A_{2n-1}} - \frac{1}{2} 1_{\cup_{j\geq 2n} A_j} = 1_{A_{2n-1}} - 1_{\cup_{j>2n} A_j} - \frac{1}{2}(1_{A_{2n}} - 1_{\cup_{j>2n} A_j}) \in X$. Thus $X_\infty \supset L^0_\infty(|\mu|)$. Lemma 1.1 implies that μ is also uniquely determined by (L). On the other hand $\frac{2}{3^n} > \frac{1}{3^n} = \sum_{i>n} \frac{2}{3^i}$ and thus for μ^+ and μ^- each atom is a bully.*

Notice that there is no restriction imposed on $\frac{d\nu}{d\mu}$ in this example and thus Theorem 3.5 does not apply. It seems likely that the restriction is not necessary in general.

4.3 An Arbitrarily Distortable Banach Space

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Abstract:
In this work we construct a Tsirelson like Banach space which is arbitrarily distortable.

4.3.1 Introduction.

We consider the following notions.

Definition 12. *Let X be an infinite dimensional Banach space, and $\|\cdot\|$ its norm. If $|\cdot|$ is an equivalent norm on X and $\lambda > 1$ we say $|\cdot|$ is a λ-distortion of X if for each infinite dimensional subspace Y of X we have*

$$\sup\left\{\frac{|y_1|}{|y_2|} : y_1, y_2 \in Y \quad \|y_1\| = \|y_2\| = 1\right\} \geq \lambda \ .$$

X is called λ-distortable if there exists a λ-distortion on X. X is called distortable if X is λ-distortable for some $\lambda > 1$, and X is called arbitrarily distortable if X is λ-distortable for all $\lambda > 1$.

Remark 45. *R.C. James[197] showed that the spaces ℓ_1 and c_0 are not distortable. Until now these are the only known spaces which are not distortable.*

From the proof of [[198], Theorem 5.2, p.145] it follows that each infinite dimensional uniform convex Banach space which does not contain a copy of ℓ_p, $1 < p < \infty$, has a distortable subspace. In [8] this result was generalized to any infinite dimensional Banach space which does not contain a copy of ℓ_p, $1 \leq p < \infty$, or c_0. A famous open problem (the "distortion problem") is the question whether or not ℓ_p, $1 < p < \infty$, is distortable. In this paper we construct a Banach space X which is arbitrarily distortable. We first want to mention the following questions which are suggested by the existence of such a space.

Problem. Is every distortable Banach space arbitrarily distortable? Is, for example, Tsirelson's space T (as presented in [[199], Example 2.e.1]) arbitrarily distortable?

4.3.2 Construction of X.

We first want to introduce some notations.

The vector space of all real valued sequences (x_n) whose elements are eventually zero is denoted by c_{00}; (e_i) denotes the usual unit vector basis of c_{00}, *i.e.*, $e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ if $i \neq j$. For $x = \sum_{i=1}^{\infty} \alpha_i e_i \in c_{00}$ the set $supp(x) = \{i \in \mathbb{N} : \alpha_i \neq 0\}$ is called the *support of* x. If E and F are two finite subsets of $\mathbb{N}$ we write $E < F$ if $\max(E) < \min(F)$, and for $x, y \in c_{00}$ we write $x < y$ if $supp(x) < supp(y)$. For $E \subset \mathbb{N}$ and $x = \sum_{i=1}^{\infty} x_i e_i \in c_{00}$ we put $E(x) := \sum_{i \in E} x_i e_i$. For the construction of X we need a function $f : [1, \infty) \to [1, \infty)$ having the properties (f_1) through (f_5) as stated in the following lemma. The verification of (f_1), (f_2), and (f_3) are trivial while the verification of (f_4) and (f_5) are straightforward.

Lemma 4.3.1. *Let $f(x) = \log_2(x+1)$, for $x \geq 1$. Then f has the following properties:*

1. (f_1) *$f(1) = 1$ and $f(x) < x$ for all $x > 1$,*
2. (f_2) *f is strictly increasing to ∞,*
3. (f_3) *$\lim_{x\to\infty} \frac{f(x)}{x^q} = 0$ for all $q > 0$,*
4. (f_4) *the function $g(x) = \frac{x}{f(x)}$, $x \geq 1$, is concave, and*
5. (f_5) *$f(x) \cdot f(y) \geq f(x \cdot y)$ for $x, y \geq 1$.*

For the sequel we fix a function f having the properties stated in Lemma 1. On c_{00} we define by induction for each $k \in \mathbb{N}_0$ a norm $|\cdot|_k$. For $x = \sum x_n \cdot e_n \in c_{00}$ let $|x|_0 = \max_{n\in\mathbb{N}} |x_n|$. Assuming that $|x|_k$ is defined for some $k \in \mathbb{N}_0$ we put

$$|x|_{k+1} = \max_{\substack{\ell\in\mathbb{N} \\ E_1<E_2<\cdots<E_\ell \\ E_i\subset\mathbb{N}}} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |E_i(x)|_k \ .$$

Since $f(1) = 1$ it follows that $(|x|_k)$ is increasing for any $x \in c_{00}$ and since $f(\ell) > 1$ for all $\ell \geq 2$ it follows that $|e_i|_k = 1$ for any $i \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Finally, we put for $x \in c_{00}$

$$\|x\| = \max_{k\in\mathbb{N}} |x|_k \ .$$

Then $\|\cdot\|$ is a norm on c_{00} and we let X be the completion of c_{00} with respect to $\|\cdot\|$.

The following proposition states some easy facts about X.

Proposition 4.3.2. *a) (e_i) is a 1-subsymmetric and 1-unconditional basis of X; i.e., for any $x = \sum_{i=1}^{\infty} x_i e_i \in X$, any strictly increasing sequence $(n_i) \subset \mathbb{N}$ and any $(\epsilon_i)_{i\in\mathbb{N}} \subset \{-1, 1\}$ it follows that*

$$\Big\| \sum_{i=1}^{\infty} x_i e_i \Big\| = \Big\| \sum_{i=1}^{\infty} \epsilon_i x_i e_{n_i} \Big\| \ .$$

b) For $x \in X$ it follows that

$$\|x\| = \max\Bigg\{ |x|_0 \ , \ \sup_{\substack{\ell\geq 2 \\ E_1<E_2<\cdots<E_\ell}} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(x)\| \Bigg\}$$

(where $|x|_0 = \sup_{n\in\mathbb{N}} |x_n|$ for $x = \sum_{i=1}^{\infty} x_i e_i \in X$).

Proof of Proposition 2. Part (a) follows from the fact that (e_i) is a 1-unconditional and 1-subsymmetric basis of the completion of c_{00} with respect to $|\cdot|_k$ for any $k \in \mathbb{N}_0$, which can be verified by induction for every $k \in \mathbb{N}$.

Since c_{00} is dense in X it is enough to show the equation in (b) for an $x \in c_{00}$. If $\|x\| = |x|_0$ it follows for all $\ell \geq 2$ and finite subsets $E_1, E_2, \ldots, E_\ell$ of $\mathbb{N}$ with $E_1 < E_2 < \cdots < E_\ell$

$$\frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(x)\| = \max_{k\geq 0} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |E_i(x)|_k \leq \max_{k\geq 1} |x|_k \leq \|x\| \ ,$$

which implies the assertion in this case.

If $\|x\| = |x|_k > |x|_{k-1} \ge |x|_0$, for some $k \ge 1$, there are $\ell, \ell' \in \mathbb{N}$, $\ell \ge 2$, finite subsets of $\mathbb{N}$, $E_1, E_2, \ldots, E_\ell$ and $E'_1, E'_2, \ldots, E'_{\ell'}$ with $E_1 < E_2 < \cdots < E_\ell$ and $E'_1 < E'_2 < \cdots < E'_{\ell'}$, and a $k' \in \mathbb{N}$ so that

$$\begin{aligned}
\|x\| &= |x|_k \\
&= \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |E_i(x)|_{k-1} \\
&\le \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(x)\| \\
&\le \sup_{\substack{2 \le \tilde{\ell} \\ widetildeE_1 < widetildeE_2 < \cdots widetildeE_{\tilde{\ell}}}} \frac{1}{f(\tilde{\ell})} \sum_{i=1}^{\tilde{\ell}} \|widetildeE_i(x)\| \\
&= \frac{1}{f(\ell')} \sum_{i=1}^{\ell'} \|E'_i(x)\| \\
&= \frac{1}{f(\ell')} \sum_{i=1}^{\ell'} |E'_i(x)|_{k'} \\
&\le |x|_{k'+1} \le \|x\| \ ,
\end{aligned}$$

which implies the assertion. □

Remark 46. *a) The equation in Proposition 2(b) determines the norm $\|\cdot\|$ in the following sense: If $|\cdot|$ is a norm on c_{00} with $|e_i| = 1$ for all $i \in \mathbb{N}$ and with the property that*

$$|x| = \max\left\{ |x|_0 \ , \sup_{\substack{\ell \ge 2 \\ E_1 < E_2 \cdots E_\ell}} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} |E_i(x)| \right\}$$

for all $x \in c_{00}$, then it follows that $\|\cdot\|$ and $|\cdot|$ are equal. Indeed one easily shows by induction for each $m \in \mathbb{N}$ and each $x \in c_{00}$ with $\#supp(x) = m$ that $\|x\| = |x|$.

b) The equation in Proposition 2(b) is similar to the equation which defines Tsirelson's space T [LT, Example 2.e.1]. Recall that T is generated by a norm $\|\cdot\|_T$ on c_{00} satisfying the equation

$$\|x\|_T = \max\left\{ |x|_0 \ , \sup_{\substack{\ell \in \mathbb{N} \\ \ell \le E_1 < \cdots E_\ell}} \frac{1}{2} \sum_{i=1}^{\ell} \|E_i(x)\|_T \right\}$$

(where $\ell \le E_1$ means that $\ell \le \min E_1$). Note that in the above equation the supremum is taken over all "admissible collections" $E_1 < E_2 < \cdots < E_\ell$ (meaning that $\ell \le E_1$) while the norm on X is computed by taking all collections $E_1 < E_2 < \cdots < E_\ell$. This forces the unit vectors in T to be not subsymmetric, unlike in X. The admissibility condition, on the other hand, is necessary in order to imply that T does not contain any ℓ_p, $1 \le p < \infty$, or c_0, which was the purpose of its construction.

We will show that X does not contain any subspace isomorphic to ℓ_p, $1 < p < \infty$, or c_0 and secondly that X is distortable, which by[197] implies that it cannot contain a copy of ℓ_1 either.

Thus, in the case of X, the fact that X does not contain a copy of ℓ_1 is caused by the factor $\frac{1}{f(\ell)}$ (replacing the constant factor $\frac{1}{2}$ in T) which decreases to zero for increasing ℓ. In order to state the main result we define for $\ell \in \mathbb{N}$, $\ell \geq 2$, and $x \in X$

$$\|x\|_\ell := \sup_{E_1<E_2<\cdots<E_\ell} \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(x)\| \ .$$

For each $\ell \in \mathbb{N}$, $\|\cdot\|_\ell$ is a norm on X and it follows that

$$\frac{1}{f(\ell)}\|x\| \leq \|x\|_\ell \leq \|x\| \ , \quad \text{for} \ \ x \in X \ .$$

Theorem 4.3.3. *For each $\ell \in \mathbb{N}$, each $\epsilon > 0$, and each infinite dimensional subspace Z of X there are $z_1, z_2 \in Z$ with $\|z_1\| = \|z_2\| = 1$ and*

$$\|z_1\|_\ell \geq 1 - \epsilon \ , \qquad \textit{and} \qquad \|z_2\|_\ell \leq \frac{1+\epsilon}{f(\ell)} \ .$$

In particular, $\|\cdot\|_\ell$ is an $f(\ell)$-distortion for each $\ell \in \mathbb{N}$.

Remark 47. *Considering for $n \in \mathbb{N}$ the space $T_{1/n}$ (see for example [200]) which is the completion of c_{00} under the norm $\|\cdot\|_{(T,1/n)}$ satisfying the equation*

$$\|x\|_{(T,1/n)} = \max\left\{ |x|_0 \ , \sup_{\ell \leq E_1<E_2\cdots E_\ell} \frac{1}{n} \cdot \sum_{i=1}^{\ell} \|E_i(x)\|_{(T,1/n)} \right\}$$

for all $x \in c_{00}$ and putting for $x \in T_{1/n}$

$$|x|_{(T,1/n)} = \sup_{E_1<E_2<\cdots<E_n} \sum_{i=1}^{n} \|E_i(x)\|_{1/n}$$

E. Odell [199] observed that $|\cdot|_{(T,1/n)}$ is a $c{\cdot}n$ distortion of $T_{1/n}$ (where c is a universal constant).

In order to show Theorem 3 we will state the following three lemmas, and leave their proof for the next section.

Lemma 4.3.4. *For $n \in \mathbb{N}$ it follows that*

$$\Big\|\sum_{i=1}^{n} e_i\Big\| = \frac{n}{f(n)} \ .$$

For the statement of the next lemma we need the following notion. If Y is a Banach space with basis (y_i) and if $1 \leq p \leq \infty$ we say that ℓ_p is finitely block represented in Y if for any $\epsilon > 0$ and any $n \in \mathbb{N}$ there is a normalized block $(z_i)_{i=1}^n$ of length n of (y_i), which is $(1+\epsilon)$-equivalent to the unit basis of ℓ_p^n and we call (z_i) a block of (y_i) if $z_i = \sum_{j=k_{i-1}+1}^{k_i} \alpha_j y_j$ for $i = 1, 2, \ldots$ and some $0 = k_0 < k_1 < \cdots$ in $\mathbb{N}_0$ and $(\alpha_j) \subset \mathbb{R}$.

Lemma 4.3.5. *ℓ_1 is finitely block represented in each infinite block of (e_i).*

Lemma 4.3.6. *Let (y_n) be a block basis of (e_i) with the following property: There is a strictly increasing sequence $(k_n) \subset \mathbb{N}$, a sequence $(\epsilon_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty} \epsilon_n = 0$ and for each n a normalized block basis $(y(n,i))_{i=1}^{k_n}$ which is $(1+\epsilon_n)$-equivalent to the $\ell_1^{k_n}$-unit basis so that*

$$y_n = \frac{1}{k_n} \sum_{i=1}^{k_n} y(n,i) \ .$$

Then it follows for all $\ell \in \mathbb{N}$

$$\lim_{n_1\to\infty} \lim_{n_2\to\infty} \dots \lim_{n_\ell\to\infty} \Big\| \sum_{i=1}^{\ell} y_{n_i} \Big\| = \frac{\ell}{f(\ell)} \ .$$

Lemma 4.3.7. *Let Z be an infinite dimensional subspace of X and $\epsilon > 0$. By passing to a further subspace and by a standard perturbation argument we can assume that Z is generated by a block of (e_i)*

Choice of z_1:

By Lemma 5 and Lemma 6 one finds $(y_i)_{i=1}^{\ell} \subset Y$, with $y_1 < y_2 < \cdots < y_\ell$ so that $\|y_i\| \geq 1-\epsilon$, $1 \leq i \leq \ell$, and so that $\| \sum_{i=1}^{\ell} y_i \| \leq \frac{\ell}{f(\ell)}$. Thus, choosing

$$z_1 = \sum_{i=1}^{\ell} y_i \Big/ \Big\| \sum_{i=1}^{\ell} y_i \Big\|$$

it follows that

$\geq 1 - \epsilon$,

which shows the desired property of z_1.

Choice of z_2:

Let $n \in \mathbb{N}$ so that $\frac{4\ell}{n} \leq \epsilon$ and choose according to Lemma 5 normalized elements $x_1 < x_2 < \cdots < x_n$ of Z so that $(x_i)_{i=1}^{n}$ is $(1+\epsilon/2)$-equivalent to the unit basis of ℓ_1^n and put

$$z_2 = \sum_{i=1}^{n} x_i \Big/ \Big\| \sum_{i=1}^{n} x_i \Big\| \ .$$

Now let $E_1, \ldots, E_\ell$ be finite subsets of $\mathbb{N}$ so that $E_1 < E_2 < \cdots < E_\ell$ and so that

$$\|z_2\|_\ell = \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|E_i(z_2)\| \ .$$

We can assume that E_i is an interval in $\mathbb{N}$ for each $i \leq \ell$. For each $i \in \mathbb{N}$ there are at most two elements $j_1, j_2 \in \{1, \ldots, n\}$ so that $supp(x_{j_s}) \cap E_i \neq \emptyset$ and $supp(x_{j_s}) \setminus E_i \neq \emptyset$, $s = 1, 2$. Putting for $i = 1, 2, \ldots, \ell$

$$widetildeE_i := \cup \big\{ supp(x_j) : j \leq n \ \text{ and } \ supp(x_j) \subset E_i \big\}$$

it follows that $\|E_i(z_2)\| \leq \|widetildeE_i(z_2)\| + \frac{2}{n}$, and, thus, from the fact that $(widetildeE_i(z_2) : i = 1, 2, \ldots, \ell)$ is a block of a sequence which is $(1+\epsilon/2)$-equivalent to the ℓ_1^n unit basis, it follows that

$$\|z_2\|_\ell \leq \frac{\ell}{2n} + \frac{1}{f(\ell)} \sum_{i=1}^{\ell} \|widetildeE_i(z_2)\| \leq \frac{\epsilon}{2} + \frac{1+\epsilon/2}{f(\ell)} \Big\| \sum_{i=1}^{\ell} widetildeE_i(z_2) \Big\| \leq \epsilon + \frac{1}{f(\ell)} \ ,$$

which verifies the desired property of z_2.

4.3.3 Proof of Lemmas 4, 5 and 6.

Proof of Lemma 4. By induction we show for each $n \in \mathbb{N}$ that $\|\sum_{i=1}^n e_i\| = \frac{n}{f(n)}$. If $n = 1$ the assertion is clear. Assume that it is true for all $\tilde{n} < n$, where $n \geq 2$. Then there is an $\ell \in \mathbb{N}$, $2 \leq \ell \leq n$, and there are finite subsets of $\mathbb{N}$, $E_1 < E_2 < \cdots < E_\ell$, so that

$$\begin{aligned}
\Big\|\sum_{i=1}^n e_i\Big\| &= \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \Big\| E_j\Big(\sum_{i=1}^n e_i\Big)\Big\| \\
&= \frac{1}{f(\ell)} \sum_{j=1}^{\ell} \frac{n_i}{f(n_i)} \qquad \text{[where } n_i = \#E_i \text{ and } \textstyle\sum n_i = n] \\
&= \frac{\ell}{f(\ell)} \sum_{j=1}^{\ell} \frac{1}{\ell} \cdot \frac{n_i}{f(n_i)} \\
&\leq \frac{\ell}{f(\ell)} \, \frac{\frac{n}{\ell}}{f(\frac{n}{\ell})} \qquad \text{[Property } (f_4) \text{ of Lemma 1]} \\
&= \frac{n}{f(\ell) \cdot f(\frac{n}{\ell})} \\
&\leq \frac{n}{f(n)} \qquad \text{[Property } (f_5) \text{ of Lemma 1]}
\end{aligned}$$

Since it is easy to see that $\|\sum_{i=1}^n e_i\| \geq \frac{n}{f(n)}$, the assertion follows. ☐

Proof of Lemma 5. The statement of Lemma 5 will essentially follow from the Theorem of Krivine ([201, 202]). It says that for each basic sequence (y_n) there ia a $1 \leq p \leq \infty$ so that ℓ_p is finitely block represented in (y_i). Thus, we have to show that ℓ_p, $1 < p \leq \infty$, is not finitely represented in any block basis of (e_i). This follows from the fact that for any $1 < p \leq \infty$, any $n \in \mathbb{N}$ and any block basis $(x_i)_{i=1}^n$ of (e_i) we have

$$\Big\|\frac{1}{n^{1/p}} \sum_{i=1}^n x_i\Big\| \geq \frac{1}{n^{1/p}} \, \frac{n}{f(n)} = \frac{n^{1-1/p}}{f(n)}$$

and from (f_3). ☐

Proof of Lemma 6. Let $y_n = \frac{1}{k_n} \sum_{i=1}^{k_n} y(n,i)$, for $n \in \mathbb{N}$ and $(y(n,i))_{i=1}^{k_n}$ $(1+\epsilon_m)$-equivalent to the $\ell_1^{k_n}$ unit basis. For $x, \tilde{x} \in c_{00}$ and $m \in \mathbb{N}$ with $x < e_m < \tilde{x}$ we will show that

$$(*) \qquad \lim_{n\to\infty} \|x + y_n + \tilde{x}^{(n)}\| = \|x + e_m + \tilde{x}\| \,,$$

where

$$\tilde{x}^{(n)} = \sum_{i=m+1}^{\infty} \tilde{x}_i \cdot e_{i+s_n} \qquad \Big(\tilde{x} = \sum_{i=m+1}^{\infty} \tilde{x}_i e_i\Big)$$

and $s_n \in \mathbb{N}$ is chosen big enough so that $y_n < \tilde{x}^{(n)}$.

This would, together with Lemma 4, imply the assertion of Lemma 6. Indeed, for $\ell \in \mathbb{N}$ it follows from $(*)$ that

$$\frac{\ell}{f(\ell)} = \Big\|\sum_{i=1}^{\ell} e_i\Big\| \qquad \text{[Lemma 4]}$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \Big\| e_1 + \sum_{i=2}^{\ell} e_{i+n} \Big\| \qquad \text{[subsymmetry]}\\
&= \lim_{n_1\to\infty} \lim_{n\to\infty} \Big\| y_{n_1} + \sum_{i=2}^{\ell} e_{i+n} \Big\|\\
&= \lim_{n_1\to\infty} \lim_{n\to\infty} \lim_{m\to\infty} \Big\| y_{n_1} + e_n + \sum_{i=3}^{\ell} e_{i+m} \Big\|\\
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \lim_{m\to\infty} \Big\| y_{n_1} + y_{n_2} + \sum_{i=3}^{\ell} e_{i+m} \Big\|\\
&\vdots\\
&= \lim_{n_1\to\infty} \lim_{n_2\to\infty} \dots \lim_{n_\ell\to\infty} \Big\| \sum_{i=1}^{\ell} y_{n_i} \Big\| .
\end{aligned}
$$

In order to prove $(*)$ we show first the following ☐

Claim. For $x, y \in c_{00}$, and $n \in \mathbb{N}$, with $x < e_n < y$ and $\alpha, \beta \in \mathbb{R}_0^+$ it follows that

$$\|x + \alpha e_n\| + \|\beta e_n + y\| \le \max\big\{\|x + (\alpha+\beta)e_n\| + \|y\|, \|x\| + \|(\alpha+\beta)e_n + y\|\big\} .$$

We show by induction for all $k \in \mathbb{N}_0$, all $x, y \in c_{00}$, and $n \in \mathbb{N}$, with $\#supp(x) + \#supp(y) \le k$, and $x < e_n < y$ and all $q_1, q_2, \alpha, \beta \in \mathbb{R}_0^+$ that

$$
\begin{aligned}
&q_1\|x + \alpha t_n\| + q_2\|\beta e_n + y\| \le\\
&\qquad \le \max\big\{q_1\|x + (\alpha+\beta)e_n\| + q_2\|y\|,\ q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\|\big\} .
\end{aligned}
$$

For $k = 0$ the assertion is trivial. Suppose it is true for some $k \ge 0$ and suppose $x, y \in c_{00}$, $x < e_n < y$ and $\#supp(x) + \#supp(y) = k + 1$. We distinguish between the following cases.

Case 1.
$\|x + \alpha e_n\| = |x + \alpha e_n|_0$ and $\|\beta e_n + y\| = |\beta e_n + y|_0$.

If $\|x + \alpha e_n\| = |x|_0$, then

$$q_1\|x + \alpha e_n\| + q_2\|\beta e_n + y\| = q_1\|x\| + q_2\|\beta e_n + y\| \le q_1\|x\| + q_2\|(\alpha+\beta)e_n + y\| .$$

If $\|\beta e_n + y\| = |y|_0$ we proceed similarly and if $\|x + \alpha e_n\| = \alpha$ and $\|\beta e_n + y\| = \beta$, and if w.l.o.g., $q_1 \le q_2$, it follows that

$$q_1\|x + \alpha e_n\| + q_2\|\beta e_n + y\| = q_1\alpha + q_2\beta \le q_2(\alpha+\beta) \le q_1\|x\| + q_2\|e_n(\alpha+\beta) + y\| .$$

Case 2.
$\|x + \alpha e_n\| \ne |x + \alpha e_n|_0$.

Then we find $\ell \ge 2$ and $E_1 < E_2 < \dots < E_\ell$ so that $E_i \cap supp(x) \ne \emptyset$ for $i = 1, \dots, \ell$ and

[By the induction hypothesis] $\le \max\big\{q_1\|x+(\alpha+\beta)e_n\|+q_2\|y\|,\ q_1\|x\|+q_2\|(\alpha+\beta)e_n+y\|\big\}$,

$$
\begin{aligned}
&q_1\|x+\alpha e_n\|+q_2\|\beta e_n+y\| \\
&\quad = \frac{q_1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell-1}\|E_i(x)\|+\|E_\ell(x+\alpha e_n)\|\Bigg]+q_2\|\beta e_n+y\| \\
&\quad \le \frac{q_1}{f(\ell)}\sum_{i=1}^{\ell-1}\|E_i(x)\|+\{ \\
&\quad \text{or} \\
&\frac{q_1}{f(\ell)}\|E_\ell(x)\|+q_2\|(\alpha+\beta)e_n+y\|
\end{aligned}
$$

[By the induction hypothesis] $\le \max\big\{q_1\|x+(\alpha+\beta)e_n\|+q_2\|y\|,\ q_1\|x\|+q_2\|(\alpha+\beta)e_n+y\|\big\}$,

which shows the assertion in this case.

In the case that $\|\beta e_n+y\| \neq |\beta e_n+y|_0$ we proceed like in Case 2.

In order to show the equation $(*)$ we first observe that for all $k \in \mathbb{N}_0$, $|x+e_m+\tilde{x}|_k \le \|x+y_n+\tilde{x}^{(n)}\|$ (which easily follows by induction for each $k \in \mathbb{N}$) and, thus, that $\liminf_{n\to\infty}\|x+y_n+\tilde{x}^{(n)}\| \ge \|x+e_m+\tilde{x}\|$. Since every subsequence of (y_n) still satisfies the assumptions of Lemma 6 it is enough to show that

$$\liminf_{n\to\infty}\|x+y_n+\tilde{x}^{(n)}\| \le \|x+e_m+\tilde{x}\| \ .$$

This inequality will be shown by induction for each $k \in \mathbb{N}_0$ and all $x < e_m < \tilde{x}$ with $\#supp(x)+\#supp(\tilde{x}) \le k$. For $k=0$ the assertion is trivial. We assume the assertion to be true for some $k \ge 0$ and we fix $x, \tilde{x} \in c_{00}$ with $x < e_m < \tilde{x}$ and $\#supp(x)+\#supp(\tilde{x}) = k+1$.

We consider the following three cases:

Case 1.
$\|x+y_n+\tilde{x}\| = |x+y_n+\tilde{x}\|_0$ for infinitely many $n \in \mathbb{N}$. Since

$$|x+y_n+\tilde{x}^{(n)}|_0 \le |x+e_m+\tilde{x}|_0 \ , \qquad n \in \mathbb{N} \ ,$$

the assertion follows.

Case 2.
For a subsequence (y'_n) of (y_n) we have

$$\|x+y'_n+\tilde{x}\| = \frac{1}{f(\ell_n)}\sum_{i=1}^{\ell_n}\|E_i^{(n)}(x+y'_n+\tilde{x})\|$$

where $\ell_n \uparrow \infty$ and $E_1^{(n)} < E_2^{(n)} < \cdots < E_{\ell_n}^{(n)}$ are finite subsets of $\mathbb{N}$. Since $f(\ell_n) \to \infty$ for $n \to \infty$ it then follows that

$$\liminf_{n\to\infty}\|x+y_n+\tilde{x}^{(n)}\| = 1 \le \|x+e_n+\tilde{x}\| \ .$$

Assume now that neither Case 1 nor Case 2 occurs. By passing to a subsequence we can assume

Case 3.
There is an $\ell \geq 2$ so that

$$\lim_{n\to\infty}\left(\|x+y_n+\tilde{x}^{(n)}\| - \frac{1}{f(\ell)}\sum_{i=1}^{\ell}\|E_i^{(n)}(x+y_n+\tilde{x}^{(n)})\|\right) = 0$$

where $E_1^{(n)} < \cdots < E_\ell^{(n)}$ are finite subsets of $\mathbb{N}$ with the following properties:

a) $supp(x+y_n+\tilde{x}^{(n)}) \cap E_i^{(n)} \neq \emptyset$, $i \leq \ell$, and

$supp(x+y_n+\tilde{x}^{(n)}) \subset \bigcup_{i=1}^{\ell} E_i^{(n)}$

b) The set $supp(x) \cap E_i^{(n)}$, $i = 1,\ldots,\ell$, does not depend on n (note that $supp(x) < \infty$), and we denote it by E_i^x.

c) There are subsets $\widetilde{E}_1 < \widetilde{E}_2 < \cdots < \widetilde{E}_\ell$ of $supp(\tilde{x})$ and integers r_n so that $supp(\tilde{x}^{(n)}) \cap E_i^{(n)} = \widetilde{E}_i + r_n$, for $n \in \mathbb{N}$, (we use the convention that $\phi < E$ for any finite $E \subset \mathbb{N}$),

d) for $i \leq \ell$ and $1 \leq j \leq k_n$ we have either $supp(y(n,j)) \subset E_i^{(n)}$ or $supp(y(n,j)) \cap E_i^{(n)} = \emptyset$.

Indeed, letting for $i \leq \ell$

$$\widetilde{E}_i^{(n)} := \begin{cases} E_i^{(n)} \ \text{ if } \ E_i^n \cap supp(y_n) = \emptyset \\ supp\big(y(n,s)\big) \cup E_i^{(n)} \setminus supp\big(y(n,t)\big) \\ \qquad \text{where } \ s := \min\big\{\tilde{s} : supp\big(y(n,\tilde{s})\big) \cap E_i^{(n)} \neq \emptyset\big\} \\ \qquad \text{and } \ t := \max\big\{\tilde{s} : supp\big(y(n,\tilde{t})\big) \cap E_i^{(n)} \neq \emptyset\big\} \end{cases}$$

the value $\sum_{i=1}^{\ell}\|E_i^{(n)}(x+y_n+\tilde{x}^{(n)})\|$ differs from $\sum_{i=1}^{\ell}\|\widetilde{E}_i^{(n)}(x+y_n+\tilde{x}^{(n)})\|$ at most by $2\ell/k_n$, which shows that d) can be assumed.

e) For $i \leq \ell$ the value

$$q_i := \lim_{n\to\infty}\frac{\#\{j \leq k_n, supp(y(n,j)) \subset E_i^{(n)}\}}{k_n}$$

exists.

Now we distinguish between the following subcases.

Case 3a. There are $\ell_1, \ell_2 \in \mathbb{N}$, so that $1 \leq \ell_1 \leq \ell_2 - 2 < \ell_2 \leq \ell$ and

$$\|x+y_n+\tilde{x}^{(n)}\| = \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1-1}\|E_i^{(n)}(x)\| + \|E_{\ell_1}^{(n)}(x+y_n)\| + \sum_{i=\ell_1+1}^{\ell_2-1}\|E_i^{(n)}(y_n)\|$$
$$+ \|E_{\ell_2}^{(n)}(y_n+\tilde{x}^{(n)})\| + \sum_{i=\ell_2+1}^{\ell}\|E_i^{(n)}(\tilde{x}^{(n)})\|\Bigg\} .$$

In this case it follows that

$$\begin{aligned}\|x+y_n+\tilde{x}^{(n)}\| &\le \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1}\|E_i^{(n)}(x)\| + \sum_{i=\ell_1}^{\ell_2}\|E_i^{(n)}(y_n)\| + \sum_{i=\ell_2}^{\ell}\|E_i^{(n)}(\tilde{x}^{(n)})\|\Bigg]\\ &\le \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1}\|E_i^{(n)}(x)\| + 1+\epsilon_n + \sum_{i=\ell_2}^{\ell}\|E_i^{(n)}(\tilde{x}^{(n)})\|\Bigg]\end{aligned}$$

[By d) and the fact that $(y(j,n))_{j=1}^{k_n}$ is $(1+\epsilon_n)$-equivalent to the $\ell_1^{k_n}$-unit basis]

$$\le \|x+e_m+\tilde{x}\| + \epsilon_n \quad ,$$

Note that

$$\big[\ell_1+1+(\ell-\ell_2+1)\le \ell\big]$$

which implies the assertion in this case.

Case 3b. There is an $1\le \ell_1\le \ell$ so that

$$\|x+y_n+\tilde{x}^{(n)}\| = \frac{1}{f(\ell)}\Bigg[\sum_{i=1}^{\ell_1-1}\|E_i^{(n)}(x)\| + \|E_{\ell_1}^{(n)}(x+y_n+\tilde{x}^{(n)})\| + \sum_{i=\ell_1+1}^{\ell}\|E_i^{(n)}(x^{(n)})\|\Bigg] .$$

Then the assertion can be deduced from the induction hypothesis (note, that by a) and the fact that $\ell\ge 2$ we have that $\#suppE_{\ell_1}^{(n)}(x+\tilde{x}^{(n)}) < \#supp(x+\tilde{x}^{(n)})$).

Case 3c. There is an $\ell_1<\ell$ so that

$$\begin{aligned}\|x+y_n+\tilde{x}^{(n)}\| = \frac{1}{f(\ell)}\Bigg[&\sum_{i=1}^{\ell_1-1}\|E_i^{(n)}(x)\| + \|E_{\ell_1}^{(n)}(x+y_n)\| + \|E_{\ell_1+1}^{(n)}(y_n+\tilde{x}^{(n)})\|\\ &+\sum_{i=\ell_1+2}^{\ell}\|E_i^{(n)}(\tilde{x}^{(n)})\|\Bigg] .\end{aligned}$$

We can assume that $supp(x)\ne 0$ and $supp(\tilde{x})\ne 0$ (otherwise we are in case 3b). If q_{ℓ_1} (as defined in e)) vanishes it follows that $\lim_{n\to\infty}\|E_{\ell_1}^{(n)}(x+y_n)\| = \|E_{\ell_1}^{x}(x)\|$. Otherwise there is a sequence $(j_n)\subset\mathbb{N}$ with $\lim_{n\to\infty} j_n=\infty$ so that

$$E_{\ell_1}^{(n)}(y_n) = \frac{1}{k_n}\sum_{j=1}^{j_n} y(n,j)$$

and so that

$$\lim_{n\to\infty}\frac{j_n}{k_n} = q_{\ell_1} > 0 .$$

Since the sequence $(E_{\ell_1}^{(n)}(y_n)/q_{\ell_1})_{n\in\mathbb{N}}$ is asymptotically equal to the sequence $(\tilde{y}_n)$ with $\tilde{y}_n := \frac{1}{j_n}\sum_{j=1}^{j_n} y(n,j)$ (note that $(\tilde{y}_n)$ satisfies the assumption of the lemma) we deduce from the induction hypothesis for some infinite $N\subset\mathbb{N}$ that

$$\lim_{\substack{n\to\infty\\ n\in N}}\|E_{\ell_1}^{(n)}(x+y_n)\| = q_{\ell_1}\lim_{n\to\infty}\left\|E_{\ell_1}^{x}\left(\frac{x}{q_{\ell_1}}\right)+\tilde{y}_n\right\|$$

$$\begin{aligned} &\le q_{\ell_1}\left\|E^x_{\ell_1}\left(\frac{x}{q_{\ell_1}}\right)+e_m\right\| \\ &= \|E^x_{\ell_1}(x)+q_{\ell_1}e_m\| \ . \end{aligned}$$

Similarly we show for some infinite $M \subset N$, that

$$\lim_{\substack{n\to\infty \\ n\in M}} \|E^{(n)}_{\ell_1+1}(y_n+\tilde{x}^{(n)})\| \le \|q_{\ell_1+1}e_m+\widetilde{E}_{\ell_1+1}(\tilde{x})\| \ .$$

From the claim at the beginning of the proof we deduce now that

$$\begin{aligned} &\liminf_{n\to\infty} \|x+y_n+\tilde{x}^{(n)}\| \\ &\qquad \le \frac{1}{f(\ell)}\left[\sum_{i=1}^{\ell_1-1}\|E^x_i(x)\|+\|E^x_{\ell_1}(x)+q_{\ell_1}e_m\|+\|q_{\ell_1+1}e_m+\widetilde{E}_{\ell_1+1}(\tilde{x})\|+\sum_{i=\ell_1+2}^{\ell}\|\widetilde{E}_i(\tilde{x})\|\right] \\ &\qquad \le \frac{1}{f(\ell)}\left[\sum_{i=1}^{\ell_1-1}\|E^x_i(x)\|+\sum_{i=\ell_1+2}^{\ell}\|\widetilde{E}_i(\tilde{x})\|\right. \\ &\qquad\qquad \left.+\max\Big\{\|E^x_{\ell_1}(x)+e_m\|+\|\widetilde{E}_{\ell_1+1}(\tilde{x})\| \ , \ \|E^x_{\ell_1}(x)\|+\|e_m+\widetilde{E}_{\ell_1+1}(\tilde{x})\|\Big\}\right] \\ &\qquad\qquad [q_{\ell_1}+q_{\ell_1+1}=1] \\ &\qquad \le \|x+e_m+\tilde{x}\| \ , \end{aligned}$$

which shows the assertion in this case and finishes the proof of the Lemma.

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